Time-varying Information Value, Information Acquisition, Learning, and Financial Decisions

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Abstract

This paper studies how the costs and benefits of information acquisition affect a firm's investment, payout and welfare under incomplete information and imperfect learning. Information gathering reduces estimation error, alleviates investment inefficiency, encourages higher payout and improves overall welfare. The cost incurred can overturn the benefit generated and distort the optimal policies. However, the degree of distortion is state-dependent as the value of information is time-varying. Information is more valuable whenever the investment environment is more volatile, the firm is more sensitive to variations in the investment opportunity set, and the marginal utility of consumption is high. The firm behaves myopically as beliefs gravitate toward extremes and the value of information fades away.

Keywords: learning, information acquisition, investment decisions, payout policy

JEL: D8, G11, G32, G35

1 Introduction

The advent of the "Big Data" era has led to an unprecedented proliferation in data collection and information acquisition. According to Curran (2019), the revenue of the U.S. financial data service providers industry reached \$15.4 billion in 2019 and is forecast to grow at an annual rate of 2% in the next five years. The increasing importance of data information, to some extent, can be attributed to the growing complexity and surging uncertainty of the financial market, which induce participants to engage in a broader scale of information search, especially through different information service intermediaries, in order to facilitate better decision making under incomplete information. Examples are abundant. Companies purchase consumer data for production and marketing purposes, institutional investors hire proxy advisory firms for research and recommendations on their votes at shareholder meetings, fund managers make portfolio decisions with the help from analyst reports, and investors invest in mutual funds to draw on funds' information advantage in picking stocks.[1] In all these contexts, information acquirers are bounded by limited information. This information incompleteness gives rises to estimation risks and necessitates learning and information acquisition.

The objective of this paper is to provide a general framework to study the motivation and influence of information acquisition. While the traditional learning literature has documented the substantial influence of estimation risks on decisions making, this paper places emphasis on the interaction between information acquisition and estimation risks. In particular, I investigate how the opportunity to acquire additional signals affects learning, how the improvement in learning from information gathering alters corporate policies and how these changes feed back into the decision to collect further information. To answer these questions, this paper develops a dynamic learning model in which a firm determines the investment and payout policies to maximize investors' life-time utility in the presence of incomplete information and imperfect learning.

[1]Extended empirical research shows that information acquisition affects different dimensions of financial decision making. For the social media channel, see Antweiler and Frank (2004), Chen et al. (2014), among others. For the analyst channel, see Jegadeesh et al. (2004), Morgan and Stocken (2003) and Huang, Zang and Zheng (2014), among others. For proxy advisory firms, see Iliev and Lowry (2015), Larcker, McCall and Ormazabal (2015), and Malenko and Shen (2016), among others. For newsletters, see Jaffe and Mahoney (1999) and Metrick (1999).

Information is incomplete in the sense that the expected rate of return of the risky asset is unknown and evolves stochastically over time. In order to determine the risky investment, the firm learns and estimates the unknown parameter from noisy realized returns. Learning is stationary in that the information from realized returns is offset by new shocks to returns. This means estimation errors persist, and such imperfect inference necessitates further information to improve the assessment of the risky asset. The firm can acquire an extra signal to improve, although not perfectly, the inference process. Using continuous-time Kalman filtering with multiple measurement equations, I show that the newly acquired signal enhances learning by providing a new information source and refining the understanding of past return predictability. The value of information can be characterized by subsequent comparison of the optimal policies under different information sets, i.e. comparing the case without extra information aid and the case with additional signals.

Lack of accurate appraisal of the risky investment opportunity leads to inefficient investment. While a time-varying investment opportunity set induces a hedging need with respect to changes in the investment environment, learning errors create an estimation-error hedging demand. More importantly, the two hedging demands are interdependent. Information acquisition alleviates investment inefficiencies by directly reducing the latter hedging demand and indirectly altering the former one as it reduces the firm's sensitivity towards changes in the investment opportunity. The paper shows that there exists a switching threshold, measured in the level of estimated rate of return, below which the firm switches its investment strategy from long to short. Without additional information aid, the firm switches earlier in order to mitigate the estimation risk from holding the asset. Information acquisition unambiguously reduces the threshold and expands the range where the firm is willing to hold the asset. I show that the efficiency improvement effect is stronger for estimations that are close to the switching threshold, where the estimated rate of return is relatively low, and the influence of estimation risk is relatively high. The extra signal is needed to guide both the sign (long or short) and the scale of the investment.

Reduction in inference error also encourages higher payout, and hence higher consumption for the investors. Improvement in investment efficiency increases the firm's ability to smooth payout intertemporally for risk-averse investors. The firm becomes more confident in the evaluation of the investment opportunity and therefore requires lower hedging demand and precautionary savings to buffer against unanticipated shocks that can cause unfavourable shifts in the future payout. Consequently, all else equal, the firm is willing to pay out more, and the investors' welfare improves as the information set expands, suggesting "the more information, the better" for decision-making bounded by incomplete information and imperfect learning. All of these are achieved under the premise of costless acquisition.[2]

I show that the cost of information acquisition undermines the efficiency-improvement role of the additional signal and reduces the payout. In states where learning is less valuable, the firm is forced to shrink investment to a level that is lower than what it would invest without the costly signal, leading to greater underinvestment. In the states where the estimation risk imposes a greater threat to the decision-making process (when the anticipated rate of return is relatively low), the benefit outweighs the cost incurred. In other words, the distortion effect, as well as the value of information acquisition, are state-dependent, although the learning contribution of the signal is stationary.

To quantify the value of information, I further examine two measures: the certainty equivalent of wealth for information (measured as a percentage of the total net worth) and the cutoff cost (the upper bound of cost, measured as a percentage of the total amount of investment); both capture the maximum cost the firm is willing to pay. All else equal, learning accuracy is more valuable when the estimation risk has greater influence, which occurs in bad times when the estimated rate of return from investment is low relative to risks embedded; for example, when beliefs are located around the switching threshold. Information is also valued more when the investment environment is more volatile. In these scenarios, a small estimation error increases the possibility of unexpected changes that could lead to unfavourable shifts in future payouts. More importantly, the marginal utility of consumption is often high in these cases, meaning an unfavourable drop in consumption has a more significant detrimental effect. As the estimations gravitate toward extremes, the firm behaves myopically as high anticipated returns make learning a second-order concern, reducing the value of information. To the best of my knowledge, the results regarding the

[2] Acquisition cost is an important consideration, especially for soft information. For example, Chen et al. (2020) use the adoption of the high-speed train in China to show that reduction in acquisition cost results in greater information production, more accurate analyst forecasts and better analyst recommendations.

time-varying value of information are original.

The model generates predictions that are relevant to various aspects of financial economics and reconciles empirical evidence (see Section 5). First, by studying the tradeoff between the costs and benefits of active information collection, the model sheds light on the active management industry and highlights the time-varying value created by information service providers (e.g. active funds). On the one hand, the model implies that a high cost can lead to 'underperformance' of active management.[3] For example, for a 12% expected rate of return, the model predicts that the maximum cost a risk averse investor is willing to pay for a signal that improves learning accuracy by 16% is around 0.7% of the total investment. Costs exceeding this boundary overtake the benefits, negating the value of information service. On the other hand, the model reveals that the beneficial effect is stronger and dominates when the investment environment is more volatile, or when the investors are more concerned about estimation risks. Hence, the ability to deliver a positive outcome when it is needed the most (e.g. in a recession) justifies the choice of active investments. Empirical research has documented the outperformance of active funds in recessions (Glode (2011), Kacperczyk, Van Nieuwerburgh and Veldkamp (2014, 2016), and others). The model thus provides an explanation for the still puzzling observation: investors engage in active investment despite persistent underperformance after costs and widely available low-cost passive alternatives.

Second, this paper also contributes to the recent discussion about how new information technologies affect private information acquisition (see Gao and Huang (2018), Goldstein, Yang and Zuo (2020), among others). My paper shows that whether public signals crowd out or encourage private information production depends on the learning path and predicts a time-varying explanatory power of public information on financial observations. The model also generates a testable prediction that private information production is countercyclical – the motivation to search for information increases (decreases) in bad (good) times. Third, the model highlights the real effect of information acquisition. It shows that a firm's investment and payout behavior reflects its confidence in the judgement of the investment opportunity. This means that measures reducing information collection barriers can miti-

[3]Note that, in this paper, performance is measured in terms of investors' welfare rather than the rate of return as in the mutual fund literature.

gate underinvestment and improve efficiency. Overall, the model generates rich implications regarding firms' or investors' financial decisions as well as the real influence of information intermediaries (e.g. brokers, active funds, consultants).

Let me conclude by briefly reviewing the related literature. This paper relates to a broad literature on learning. This literature includes papers that examine how estimation risk affects an investor's asset allocation decisions (e.g. Williams (1977), Detemple (1986), Gennotte (1986), Brennan (1998), Kan and Zhou (2007) and others), and how the predictability of stock returns influence the dynamics of portfolio choices, asset prices and volatility (e.g. Barberis (2000), Xia (2001), Brandt et al. (2005), Johannes, Korteweg and Polson (2014) among others).[4] My paper differs from these studies in the following aspects. First, I focus on the influence of information acquisition on learning, investment, payout and investors' welfare, rather than the estimation risk itself. Instead of studying how incomplete information affects financial decisions, this paper examines the roles of information-gathering under incomplete information and persistent imperfect learning. Second, as distinct from Barberis (2000) and Xia (2001), who study the relationship between predictability, investment horizon, and optimal portfolio choices, this paper refrains from the horizon effect by incorporating stationary learning in an infinite-time setup. Such a setup allows us to focus on the long term value of information services and study the demand for information intermediaries.

More broadly, this paper is related to the literature that studies how learning influences investment timing in a real-option framework.[5] For example, Décamps, Mariotti,

[4]Note that this strand of literature adopts a traditional Bayesian approach to address estimation errors and information updating. This means investors are ambiguity-neutral. Garlappi, Uppal and Wang (2006) study the influence of estimation errors on portfolio choices by augmenting a static mean-variance portfolio model with multiple priors and ambiguity aversion.

[5]The paper also links to the literature on real options under incomplete or asymmetric information, e.g. Grenadier (1999), Lambrecht and Perraudin (2003), Hsu and Lambrecht (2007), Grenadier and Malenko (2011) among others. These papers study the exercise of an option in different contexts of asymmetric information among participants. Learning plays roles in extracting private information from the exercise strategy of the other party, which, in turn, feed back into the timing decisions of the participants. In particular, Grenadier (1999) studies how revealed information from the observed exercise decisions of peers affects the dynamic equilibrium. He also explores the value of information services and shows that it is a function of the value of the underlying asset and the signal. My paper focuses explicitly on the demand side and assumes the supply side of the information is exogenously given. Learning takes place to extract information from signals, and the paper places emphasis on how the motivation of information collection varies over time.

and Villeneuve (2005) and Klein (2008) find that with incomplete information, the optimal exercising strategy is path-dependent. Grenadier and Malenko (2010) study the influence of learning on investment timing when a firm is uncertain about both future and past shocks. While the firm is uncertain about future shocks, it also fails to fully learn about past shocks, which induces it to update beliefs as time goes by. Therefore, the real-option value consists of value to delay and value to learn. Daley, Geelen and Green (2020) model due diligence after the seller accept the acquirer's bid as a real option problem. The buyer has the right to acquire further information and decide whether to go ahead with the deal. Compared to no information acquisition, due diligence enhances the total surplus as well as the seller's payoff. Unlike these papers, which focus on the timing of investment, this paper emphasizes how the learning dynamics influence the intensity of investment in a canonical investment-consumption framework and examines the dynamic interaction of information acquisition, investment and payout. In this paper, the firm's understanding of the past return predictability is imperfect, and inference errors persist. The newly acquired signal improves the understanding of predictability of past shocks. I show that the valuation of the risky investment opportunity encapsulates the value of learning and the value of hedging. The optimal investment strategy is state-dependent and, more interestingly, the influence of information on the optimal policies is also time-varying.

The paper also links to the literature that studies dynamic corporate financial policies featured with stationary learning (e.g. Acharya and Lambrecht (2015), DeMarzo and Sannikov (2017) and He et al. (2017)). In particular, Acharya and Lambrecht (2015) build a model to study income and payout smoothing when the insiders know more than the outsiders about the firm's marginal cost. Outsiders can only learn from a noisy latent signal – sales. With the discrete Kalman filter approach, they show that incomplete inference induces the manager to manipulate outsiders' expectation by distorting the production decision (real smoothing). In my paper, with the continuous-time Kalman filter, I study the dynamics of investment and payout policies. Different from the above papers, the agent maximizes the investors' life-time utility; thus, there is no asymmetric information between insiders and outsiders. I focus on the demand side of information, and take the supply side as exogenously given. By comparing the policies under different information sets, the paper characterizes the value of information and learning, and their influence on the interaction of financial policies.

Finally, a well-established strand of papers studies the implications of learning on asset prices. For example, Wang (1993) considers an equilibrium model in which uninformed investors and informed investors trade against each other. Uninformed investors obtain information from the equilibrium price and dividends. Veronesi (2000) and Li (2005) study how information quality (i.e. the precision of signals) affects the market risk premium and volatility. Other related papers include Barlevy and Veronesi (2000), Peng (2005), Jagannathan and Liu (2019) and Jagannathan, Liu and Zhang (2019), among others. Pastor and Veronesi (2009) provide a detailed review of this area. These papers focus on the equilibrium features, and the price of information is determined through trading among agents. Diverging from these papers, I focus on how information affects the learning curve and the corresponding corporate finance implications in a partial equilibrium model by assuming the informativeness and cost of the extra signal are exogenously given.

2 The Model

The objective of this paper is to study the influence of information acquisition on the acquirer's financial decisions. Toward this end, I present a general dynamic model of learning with the choice to acquire additional signals. The analysis is illustrated in the context of a firm's payout and investment decisions in risky assets. I leave the discussion of applications and predictions in a wider context for Section 5.

2.1 Investment opportunities

Time is continuous, and the horizon is infinite. A firm[6] can invest an amount A_t in a risky asset that generates a return given by a stochastic process

$$\frac{dA_t}{A_t} = \theta_t dt + \sigma_A dZ_{At} \tag{1}$$

where σ_A is a known positive constant and Z_A is a standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The expected rate of return of the asset, θ_t, is unobservable and is governed by a random process

$$d\theta_t = (a_0 + a_1\theta_t)dt + \sigma_\theta dZ_{\theta t} = a_1(\frac{a_0}{a_1} + \theta_t)dt + \sigma_\theta dZ_\theta \equiv -\lambda_\theta(\theta_t - \overline{\theta})dt + \sigma_\theta dZ_\theta \tag{2}$$

where a_0, a_1 and σ_θ are known constants.[7] Z_θ is another standard Brownian motion with $cor(dZ_\theta, dZ_A) = \rho_{\theta A}$. The paper focuses on incomplete information and imperfect learning and abstains from other frictions such as short selling constraints. Thus, the firm can either go long the risky asset or take a short position in it. The firm finances the investment with equity W_t and net debt D_t, and can borrow and save at the risk-free rate r, i.e.

$$dD_t = rD_t dt \tag{3}$$

A negative value for D_t means that the firm has a net surplus of safe, liquid assets.

[6] The firm is run by a manager acting on behalf of the risk averse investors (shareholders). The manager can also be an owner-manager of a firm; in this case, the manager herself is risk averse and acts to maximize her lifetime utility from the firm's payout. The analysis also applies to other scenarios such as portfolio management. In this case, one could also think of the firm as a fund, where the fund manager determines the portfolio choices and payout to maximize the investors' utility. In the paper, I use firm, agent and manager interchangeably, and investors and shareholders interchangeably.

[7] With $a_1 = 0$, the process for θ is an arithmetic Brownian motion, while with $a_0 \neq 0$ and $a_1 < 0$, it is an Ornstein–Uhlenbeck mean-reverting process. In both cases, the drift evolves continuously over $\mathbb{R}$. The main numerical analysis of the paper is based on the arithmetic Brownian motion assumption. The propositions and proofs hold for a general a_1, unless stated otherwise. In unreported analysis, I also replicate the study for the case with an O-U process. The main conclusions are consistent with the arithmetic Brownian motion case.

2.2 Signals and information

The endowed information set is $\mathcal{I}_{0t} = \{A_t\}$. Without any other information, the agent can only learn from the realized returns up to time t. The agent can acquire an additional signal S_t, which expands the information set to $\mathcal{I}_{1t} = \{A_t, S_t\}$.[8] The signal evolves according to

$$dS_t = b_1 \theta_t dt + \sigma_S dZ_{St} \tag{4}$$

where b_1 and σ_S are known constants. Z_S is a Brownian motion with $cor(dZ_A, dZ_S) = \rho_{AS}$ and $cor(dZ_\theta, dZ_S) = \rho_{\theta S}$. The covariance terms are defined accordingly, e.g. $\sigma_{\theta A} \equiv \rho_{\theta A}\sigma_\theta\sigma_A$. The actual information is summarized by the filtration $\mathcal{F}^{\mathcal{I}_0}$ and $\mathcal{F}^{\mathcal{I}_1}$ defined on the information set $\mathcal{I}_0$ and $\mathcal{I}_1$, respectively. The model can be generalized to cases with multiple signals and assets, and the analysis can be adapted correspondingly. However, consideration of more than one signal or asset does not generate new economic insights but complicates the notation.[9] For simplicity's sake, I focus on the single risky asset and one signal setting.

2.3 Inference process

At each instant of time, the firm determines the risky investment and payout conditional on information up to time t. As shown in Williams (1977), Detemple (1986) and Gennotte (1986), the agent's decision-making process can be decomposed into an inference problem in which she forms an estimate of the unobservable state variable and a subsequent optimization problem in which optimal policies are set with the estimate. This two-stage optimization process is known as the separation theorem in which information and learning play crucial roles.[10]

[8] For the moment, I assume the signal is costless. This assumption is relaxed in Section 4 where the cost effect of information acquisition is taken into account. The signal can take a variety of forms in reality (for example, hiring an investment advisor, conducting market research, and observing public announcements).

[9] With multiple risky assets, the risky asset can be thought of as a composite asset (ETFs).

[10] According to Gennotte (1986), the continuous-time separation theorem holds here for the following two reasons. First, the rate of return depends linearly on θ_t. Second, with the assumption that the prior belief of θ_0 is Gaussian, the variables θ_t, A_t, and S_t, given by (1), (2), and (4) respectively, are also conditionally Gaussian, meaning that only the first two moments matter for the agent's inference and optimization.

Conditional on the information set $\mathcal{I}_i$, $i = 0$ or 1, the agent's posterior belief of θ_t is summarized by the first two moments: $m_t^{\mathcal{I}_i} \equiv \mathbb{E}(\theta_t | \mathcal{F}_t^{\mathcal{I}_i})$ and $v_t^{\mathcal{I}_i} \equiv \mathbb{E}[(\theta_t - m_t)^2 \,|\, \mathcal{F}_t^{\mathcal{I}_i}]$. The prior belief of θ is assumed to be normally distributed with mean m_0 and variance v_0; thus, the posterior estimate is also normally distributed. Applying the continuous filtering method (Liptser and Shiryaev (2001a, 2001b)), the inference process can be summarized by the following proposition:[11]

Proposition 1 *The updating rules for the conditional mean with respect to the filtration* $\mathcal{F}_t^{\mathcal{I}_0}$ *and* $\mathcal{F}_t^{\mathcal{I}_1}$ *are*

$$dm_t^{\mathcal{I}_0} = \underbrace{(a_0 + a_1 m_t^{\mathcal{I}_0})dt}_{\text{expected change}} + \underbrace{\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_0}}{\sigma_A}}_{\text{weight on the unexpected shock}} \underbrace{\frac{1}{\sigma_A}\Big(\frac{dA_t}{A_t} - m_t^{\mathcal{I}_0} dt\Big)}_{\text{surprise from the realized return}} \tag{5}$$

$$\equiv (a_0 + a_1 m_t^{\mathcal{I}_0})dt + \frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_0}}{\sigma_A} d\hat{Z}_{At}^{\mathcal{I}_0} \equiv (a_0 + a_1 m_t^{\mathcal{I}_0})dt + \xi_0 d\hat{Z}_{At}^{\mathcal{I}_0} \tag{6}$$

$$dm_t^{\mathcal{I}_1} \equiv \underbrace{(a_0 + a_1 m_t^{\mathcal{I}_1})dt}_{\text{expected change}} + \xi_A \underbrace{\frac{1}{\sigma_A}\Big(\frac{dA_t}{A_t} - m_t^{\mathcal{I}_1} dt\Big)}_{\text{surprise from the realized return}} + \xi_S \underbrace{\frac{1}{\sigma_S}\Big(dS_t - b_1 m_t^{\mathcal{I}_1} dt\Big)}_{\text{surprise from the signal}}$$

$$\equiv (a_0 + a_1 m_t^{\mathcal{I}_1})dt + \xi_A d\hat{Z}_{At}^{\mathcal{I}_1} + \xi_S d\hat{Z}_{St} \tag{7}$$

where

$$\xi_0 = \frac{\overline{v}^{\mathcal{I}_0} + \sigma_{\theta A}}{\sigma_A}, \ \xi_A \equiv \frac{\xi_1 - \rho_{AS}\xi_2}{1 - \rho_{AS}^2} \quad \text{and} \quad \xi_S \equiv \frac{\xi_2 - \rho_{AS}\xi_1}{1 - \rho_{AS}^2} \tag{8}$$

$$\text{with} \quad \xi_1 \equiv \frac{\overline{v}^{\mathcal{I}_1} + \sigma_{\theta A}}{\sigma_A} \quad \text{and} \quad \xi_2 = \frac{b_1 \overline{v}^{\mathcal{I}_1} + \sigma_{\theta S}}{\sigma_S} \tag{9}$$

$\overline{v}^{\mathcal{I}_0}$ *and* $\overline{v}^{\mathcal{I}_0}$ *denote the stationary estimation variances (errors) that are derived by solving the Riccati-type variance updating equations*[12]

$$\frac{dv_t^{\mathcal{I}_0}}{dt} = \underbrace{2a_1 v_t^{\mathcal{I}_0} + \sigma_\theta^2}_{\text{information noise}} - \underbrace{\Big(\frac{\sigma_{\theta A} + v_t^{\mathcal{I}_0}}{\sigma_A}\Big)^2}_{\text{information gain}} = 0 \tag{10}$$

$$\frac{dv_t^{\mathcal{I}_1}}{dt} = \underbrace{2a_1 v_t^{\mathcal{I}_1} + \sigma_\theta^2}_{\text{information noise}} - \underbrace{(\zeta_{At}^2 + \zeta_{St}^2 + 2\rho_{AS}\zeta_{At}\zeta_{St})}_{\text{information gain}} = 0 \tag{11}$$

[11] All proofs are in the Appendix.

[12] ζ_{At} and ζ_{St} are defined in (41) in the Appendix.

The expressions of $\overline{v}^{\mathcal{I}_0}$ *and* $\overline{v}^{\mathcal{I}_0}$ *are given by Equations* (43) *and* (44) *in the Appendix.*

Equation (6) and (7) suggest that the updating rules consist of two elements: an ex-ante deterministic component and an ex-post "surprise" component. The former represents the adjustment before any observations, while the latter describes the update after observing "surprises". $d\hat{Z}_{At}^{\mathcal{I}_0}$, $d\hat{Z}_{At}^{\mathcal{I}_1}$ and $d\hat{Z}_{St}$ are the Kalman gains capturingw the unexpected shocks. ξ_0, ξ_A and ξ_S are the weights that determine how much new information is incorporated and depend on the degree of parameter uncertainty and estimation inaccuracy. Specifically, ξ_0 implies that for $\sigma_{\theta A} > (<) \, 0$, the agent, due to the estimation error, overattributes (underattributes) a positive surprise to the increase (decrease) in the drift term, and thus overestimates the future expected return. Notice that with stationary learning, $\xi_0 = \sigma_\theta$, the agent attributes the entire unexpected jump in the realized return to a permanent shift in the drift term. Therefore, the agent, without any other information aid, behaves as if past return is a perfect predictor for future return even though it is not.

Learning is assumed to be stationary in order to focus on the cost and value of information acquisition.[13] In other words, this paper is interested in the steady-state where no further refinement of the estimate can be achieved after exhausting all the information. Focusing on the steady-state allows us to study the long-term value of information acquisition. Moreover, the stationary learning assumption is consistent with the fact that imperfect learning is not merely a transitory problem. It persists as past realizations are usually insufficient to predict future expected returns. Mathematically, the steady-state learning errors are obtained when information gain offsets the noise, i.e. the solutions to (10) and (11).[14] Examing the estimation error shows

Proposition 2 *With stationary learning and* $a_1 = 0$*, the extra signal improves the learning accuracy, that is,* $\overline{v}^{\mathcal{I}_0} \geq \overline{v}^{\mathcal{I}_1}$*.*[15]

This is rather intuitive – "the more, the better". The acquisition of new signals enables the agent to appraisal the investment opportunity more accurately. The degree of improvement

[13] The time before the stationary state is reached is not studied. Perfect learning is not ruled out, and can be achieved if a specific information set is obtained.

[14] This gives $v_0 = v_{it} = \overline{v_i}$ for all t, and $\overline{v_i}$ is determined by the information structure $\mathcal{I}_i$.

[15] For $a_1 \neq 0$, I show numerically that $\overline{v}^{\mathcal{I}_0} \geq \overline{v}^{\mathcal{I}_1}$ also holds.

in inference depends crucially on the structure of the signal. Comparing the two updating rules, one can see that information acquisition influences the learning process through two channels. First, it provides a new information source to learn, as represented by $d\hat{Z}_{St}$. Second, it refines the understanding of the past return predictability, shown by changing the weight from ξ_0 to ξ_A. Without additional information, the firm observes a return but fails to identify its properties and its exact role in predicting future returns. The new signal helps to refine the part of learning that is based on realized return and updates the belief about past shocks' contribution to future shocks. Perfect learning is still possible if a particular information structure is defined. If the drift is deterministic ($\sigma_\theta = 0$), perfect inference is always achieved ($\overline{v} = 0$) under an infinite learning horizon. However, with $\sigma_\theta \neq 0$, perfect learning is achieved if

$$\rho_{\theta A} = \pm 1 \text{ for } \mathcal{I} = \mathcal{I}_0 \tag{12}$$

$$\rho_{\theta S} = \pm 1 \quad \text{or} \quad 1 - \rho_{\theta A}^2 = (\rho_{AS} - \rho_{\theta S})^2 + 2\rho_{AS}\rho_{\theta S}(1 - \rho_{\theta A}) \text{ for } \mathcal{I} = \mathcal{I}_1 \tag{13}$$

$\rho_{\theta A} = \pm 1$ means that the past return is a sufficient statistic for the future expected rate of return as they move simultaneously and proportionally. If $\rho_{\theta A} \neq \pm 1$, then past return is a noisy signal and gives rise to learning uncertainty. Nevertheless, as can be seen from condition (13), with information acquisition, perfect learning is still possible when $\rho_{\theta A} \neq \pm 1$ as long as the extra signal acquired fills the information gap.

The processes of the return and the signal can now be rewritten in terms of variables that are fully known to the agent, $\{m^{\mathcal{I}_i}, d\hat{Z}_{At}^{\mathcal{I}_0}, d\hat{Z}_{At}^{\mathcal{I}_1}, d\hat{Z}_{St}\}$, instead of the unobservable ones, $\{\theta, dZ_{At}\}$. Conditional on $\mathcal{F}_t^{\mathcal{I}_i}$, for $i = 0$ or 1,

$$\frac{dA_t}{A_t} = m_t^{\mathcal{I}_i} dt + \sigma_A d\hat{Z}_{At}^{\mathcal{I}_i} \quad \text{and} \quad dS_t = b_1 m_t^{\mathcal{I}_1} dt + \sigma_S d\hat{Z}_{St} \tag{14}$$

2.4 The firm's decision problem

At each point in time, the firm invests an amount A_t in the risky asset and pays out c_t to the investors, given the information up to t and the firm's net wealth W_t. The net wealth process based on different information structures is given by

$$dW_t = dA_t - dB_t - c_t dt = [(r + (m_t^{\mathcal{I}_i} - r)\omega_t)W_t - c_t]dt + \omega_t \sigma_A W_t d\hat{Z}_{At}^{\mathcal{I}_i} \quad (15)$$

where $\omega_t \equiv \frac{A_t}{W_t}$ represents the proportion of the net worth invested in the risky asset. The risk averse investors have a common CRRA utility function, i.e. constant coefficient of relative risk aversion, and maximize the expected life-time utility from the payout c_t. Therefore, at each instant t, conditional on the available information, the agent optimizes by choosing the payout c_t and the investment ω_t, that is,

$$J_i(W_t, m_t) = \max_{\{c_t, \omega_t\}} \mathbb{E}_t \Big[\int_t^\infty e^{-\rho t} \frac{c_t^{1-\gamma} - 1}{1-\gamma} dt \mid \mathcal{F}_t^{\mathcal{I}_i} \Big] \quad (16)$$

where ρ is the subjective discount factor, and $\gamma > 1$ is the coefficient of relative risk aversion.[16] Thus, the two optimization problems corresponding to the two information structures are

Problem 0 (Endowed Information: 0-signal) *With filtration $\mathcal{F}^{\mathcal{I}_0}$, the agent performs the maximization as in* (16)*, subject to the intertemporal budget constraint* (15) *as well as the standard transversality condition.*

Problem 1 (Expanded Information: 1-signal) *With filtration $\mathcal{F}^{\mathcal{I}_1}$, the agent performs the maximization as in* (16)*, subject to the intertemporal budget constraint* (15) *as well as the standard transversality condition.*

As a useful benchmark, I also consider a baseline model absent of incomplete information, in which the agent is able to observe θ_t perfectly, i.e. $\mathcal{F}_t^{\mathcal{I}_2} \equiv \mathcal{F}_t$. The information structure implies $\mathcal{F}_t^{\mathcal{I}_0} \subset \mathcal{F}_t^{\mathcal{I}_1} \subset \mathcal{F}_t^{\mathcal{I}_2}$ for all t.

[16] The transversality and feasibility conditions require $\gamma > 1$. The $\gamma = 1$ (log utility) case is considered as the myopic case where the investor's optimal policies are independent of the estimation risk (Brennan (1998), Feldmand (1992)). For $\gamma < 1$, the transversality condition is violated. The stochastic expected rate of return means that θ_t can become extremely positive or negative and low risk aversion induces the agent to invest heavily into the risky asset, which makes the claims unbounded and the transversality condition violated.

Problem 2 (Perfect Information) *With filtration $\mathcal{F}_t^2$, the agent performs the maximization as in* (16) *with m_t being replaced by the true value θ_t, subject to the intertemporal constraint*

$$dW = [(r + (\theta_t - r)\omega_t)W_t - c_t]dt + \omega_t \sigma_A W_t dZ_{At} \tag{17}$$

as well as the standard transversality condition.

Frictions such as borrowing constraint and short selling restriction are not considered within this paper. This means the firm can undertake a short position in the risky asset when the estimated return is below some thresholds.

3 Optimal Policies

The expected rate of return varies over time, meaning the investment opportunity set is time-changing. Meanwhile, limited information impedes an accurate evaluation of the asset. The two intertwined uncertainties shape the firm's optimal policies. The power utility function implies the indirect utility function $J_i(W_t, m_t^{\mathcal{I}_i})$ is separable and can be written as

$$J_i(W_t, m_t^{\mathcal{I}_i}) = \frac{W_t^{1-\gamma}}{1-\gamma} G_i(m_t^{\mathcal{I}_i}) - \frac{1}{\rho}\frac{1}{1-\gamma} \equiv W_t^{1-\gamma} g_i(m_t^{\mathcal{I}_i}) - \frac{1}{\rho}\frac{1}{1-\gamma} \quad \text{for} \quad i \in \{0,1\} \tag{18}$$

where $g_i(m)$ is a function capturing how valuable the investment opportunity is for the firm and encapsulating the two sources of uncertainties and the influence of information. The propositions below present the solutions to the three optimization problems.[17]

Proposition 3 *In Problem 0, the investor's value function can be written as in* (18)*, where $g_0(m)$ exists and is continuous, finite and concave for $m_t \in \mathbb{R}$. The optimal investment policy ω_0^* and consumption policy c_0^* are:*

$$\omega_0^* = \frac{m - r}{\gamma \sigma_A^2} + \left[\frac{\sigma_{\theta A}}{\gamma \sigma_A^2}\frac{g_0'(m)}{g_0(m)} + \frac{\overline{v}^{\mathcal{I}_0}}{\gamma \sigma_A^2}\frac{g_0'(m)}{g_0(m)}\right] \equiv \omega_0^m + [\omega_0^u + \omega_0^v] \tag{19}$$

$$c_0^* = W(1-\gamma)^{-\frac{1}{\gamma}} g_0(m)^{-\frac{1}{\gamma}} \tag{20}$$

[17] The time index t and the superscript of m_t are omitted from now for expositional purposes. m refers to the estimation of the drift term with the corresponding information structure specified in the propositions.

where the function $g_0(m)$ *is the solution to*

$$\underbrace{\Big(r(1-\gamma)-\rho+\frac{1}{2}\frac{1-\gamma}{\gamma}\frac{(m-r)^2}{\sigma_A^2}\Big)g_0(m)+\frac{\gamma}{1-\gamma}(1-\gamma)^{1-\frac{1}{\gamma}}g_0(m)^{1-\frac{1}{\gamma}}}_{\text{investment factor}}$$
$$+\underbrace{\Big(a_0+a_1m+\frac{1-\gamma}{\gamma}\frac{m-r}{\sigma_A}\frac{\sigma_{\theta A}+\overline{v}^{\mathcal{I}_0}}{\sigma_A}\Big)g_0'(m)+\frac{1}{2}\frac{1-\gamma}{\gamma}\Big(\frac{\sigma_{\theta A}+\overline{v}^{\mathcal{I}_0}}{\sigma_A}\Big)^2\frac{g_0'(m)^2}{g_0(m)}}_{\text{hedging factor}} \quad (21)$$
$$+\underbrace{\frac{1}{2}\Big(\frac{\sigma_{\theta A}+\overline{v}^{\mathcal{I}_0}}{\sigma_A}\Big)^2 g_0''(m)}_{\text{information factor}}=0$$

with the boundary conditions $\lim_{m\to-\infty}g_0'(m)=\lim_{m\to+\infty}g_0'(m)=0$.

Proposition 4 *In Problem 1, the investor's value function can be written as in* (18)*, where* $g_1(m)$ *exists and is continuous, finite and concave for* $m\in\mathbb{R}$*. The optimal investment policy* ω_1^* *and consumption policy* c_1^* *are:*

$$\omega_1^*=\frac{m-r}{\gamma\sigma_A^2}+\Big[\frac{\sigma_{\theta A}}{\gamma\sigma_A^2}\frac{g_1'(m)}{g_1(m)}+\frac{\overline{v}^{\mathcal{I}_1}}{\gamma\sigma_A^2}\frac{g_1'(m)}{g_1(m)}\Big]\equiv\omega_1^m+[\omega_1^u+\omega_1^v] \quad (22)$$
$$c_1^*=W(1-\gamma)^{-\frac{1}{\gamma}}g_1(m)^{-\frac{1}{\gamma}} \quad (23)$$

where the function $g_1(m)$ *is the solution to*

$$\underbrace{\Big(r(1-\gamma)-\rho+\frac{1}{2}\frac{1-\gamma}{\gamma}\frac{(m-r)^2}{\sigma_A^2}\Big)g_1(m)+\frac{\gamma}{1-\gamma}(1-\gamma)^{1-\frac{1}{\gamma}}g_1(m)^{1-\frac{1}{\gamma}}}_{\text{investment factor}}$$
$$+\underbrace{\Big(a_0+a_1m+\frac{1-\gamma}{\gamma}\frac{(m-r)}{\sigma_A}\frac{\sigma_{\theta A}+\overline{v}^{\mathcal{I}_1}}{\sigma_A}\Big)g_1'(m)+\frac{1}{2}\frac{1-\gamma}{\gamma}\Big(\frac{\sigma_{\theta A}+\overline{v}^{\mathcal{I}_1}}{\sigma_A}\Big)^2\frac{g_1'(m)^2}{g_1(m)}}_{\text{hedging factor}}$$
$$+\underbrace{\frac{1}{2}\Big(\xi_A^2+\xi_S^2+2\rho_{AS}\xi_A\xi_S\Big)g_1''(m)}_{\text{information factor}}=0 \quad (24)$$

with the boundary conditions $\lim_{m\to-\infty}g_1'(m)=\lim_{m\to+\infty}g_1'(m)=0$. ξ_A *and* ξ_S *are defined as in* (8).

Proposition 5 *In Problem 2, with perfect information,* $m_t=\theta_t$ *for all* t*. The investor's value function can be written as in* (18)*, where* $g_2(\theta)$ *exists and is continuous, finite and* *concave for $\theta_t \in \mathbb{R}$. The optimal investment policy ω_2^* and consumption policy c_2^* are:*

$$\omega_2^* = \frac{\theta - r}{\gamma\sigma_A^2} + \frac{\sigma_{\theta A}}{\gamma\sigma_A^2}\frac{g_2'(\theta)}{g_2(\theta)} \equiv \omega_2^m + \omega_2^u \tag{25}$$

$$c_2^* = W(1-\gamma)^{-\frac{1}{\gamma}} g_2(\theta)^{-\frac{1}{\gamma}} \tag{26}$$

where the function $g_2(m)$ satisfies the following

$$\begin{aligned}
&\Big(r(1-\gamma) - \rho + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{(\theta - r)^2}{\sigma_A^2}\Big) g_2(\theta) + \frac{\gamma}{1-\gamma}(1-\gamma)^{1-\frac{1}{\gamma}} g_2(\theta)^{1-\frac{1}{\gamma}} \\
&+ \Big(a_0 + a_1\theta + \frac{1-\gamma}{\gamma}\frac{\theta - r}{\sigma_A}\frac{\sigma_{\theta A}}{\sigma_A}\Big) g_2'(\theta) + \frac{1}{2}\frac{1-\gamma}{\gamma}\Big(\frac{\sigma_{\theta A}}{\sigma_A}\Big)^2 \frac{g_2'(\theta)^2}{g_2(\theta)} \\
&+ \frac{1}{2}\Big(\frac{\sigma_{\theta A}}{\sigma_A}\Big)^2 g_2''(\theta) = 0
\end{aligned} \tag{27}$$

with the boundary conditions $\lim_{m\to-\infty} g_2'(\theta) = \lim_{m\to+\infty} g_2'(\theta) = 0$.[18]

If $\sigma_\theta = 0$ (then $\sigma_{\theta A} = 0$), Problem 0 and 1 are equivalent to Problem 2. As shown in the three differential equations, three factors jointly determine the valuation of the investment opportunity, for a given level of total net worth. First, the investment factor: the firm faces a larger investment opportunity set as the risky asset expands the opportunity set beyond a single riskless asset.[19] Second, the hedging factor: the firm is now exposed to a time-varying investment environment but, at the same time, can hedge against such uncertainty through the investment strategy. Third, the information factor: the attractiveness of the risky investment depends on the agent's belief about future return as well as the accuracy of the belief. Both depend on the information set. The agent can learn from the past realized return (i.e. passive learning by investing) and the extra signals (i.e. active learning by acquiring information). The influence of information acquisition features in the information channel explicitly and hedging channel implicitly by altering the firm's sensitivity to changes

[18]The optimality conditions require joint concavity of the indirect utility function (18) in both state variables W_t and m_t. This means $J_{im} > 0$ and $J_{imm} < 0$. Also, as the agent's beliefs become extreme, the agent's behaviour converges to that in the deterministic-drift scenario as in Merton (1969, 1971). Thus, the first myopic term in the investment policy dominates, and the first derivative of $g_i(m)$ for $i \in \{0, 1, 2\}$ goes to zero as $m \longrightarrow \pm\infty$

[19]Note that these two terms are identical to the only terms in the case where the drift is a constant. When the drift is constant, learning is perfect under infinite horizon, and $g(\theta)$ satisfies Equation (56) in the Appendix. It means that this factor calculates what would be the value of the risky asset for the investor as if the expected rate of return is a constant equal to m.

in the investment opportunity. Hence, the value of the investment opportunity depends on the risk-adjusted return it can generate, the embedded uncertainty, and the required learning effort.

Closed-form solutions to the problems are not available, and a standard numerical approach is used. Parameter values are given in Table 1.[20] The aggregate hedging demand $\omega_i^h \equiv \omega_i^u + \omega_i^v$, payout yield c_i/W and valuation of the risky opportunity set g_i, as well as the corresponding comparisons between different information sets are plotted in Figure 1 as functions of m.[21]

Table 1: Baseline parameter values for numerical analysis

Process	Parameter	Notation	Value
Investment opportunity	risky investment volatility	σ_A	0.14
	drift of θ_t	a_0	0.09
		a_1	0
	volatility of θ_t	σ_θ	0.05
The signal	drift of S	b_1	0.084
	volatility of S	σ_S	0.15
Correlation	$cor(d\theta, \frac{dA}{A})$	$\rho_{\theta A}$	-0.1
	$cor(d\theta, \frac{dS}{S})$	$\rho_{\theta S}$	0.5
	$cor(\frac{dA}{A}, \frac{dS}{S})$	ρ_{AS}	0.01
Other parameters	risk-free rate	r	0.05
	time discount rate	ρ	0.1
	coefficient of RRA	γ	2

3.1 Investment policy

The optimal investment decision, given in (19) and (22), has three components. ω_i^m is the familiar mean-variance term that is often quoted as the myopic investment policy. The

[20]The purpose of the numerical analysis is to demonstrate the qualitative relation between optimal decisions and information. In the figures and tables discussed in the papers, a consistent set of parameter values is used. In the numerical example, I set the risk-free rate $r = 5\%$ to match the average three-month Treasury Bill rate. The volatility of the risky asset return $\sigma_A = 0.14$, which is estimated in Xia (2001). The coefficient of RRA $\gamma = 2$ because the commonly accepted range for RRA is between 1 and 3. Sometimes in the literature, a wider range of estimates, from 0.2 to 10 or higher, is obtained (see Chetty (2006), Campo et al. (2011), and others). Other parameters characterize the informativeness of the information set. I vary the parameter values within a wide range and discover the most robust features. Unless otherwise stated, all proceeding study adopts these parameter values.

[21]In comparisons of the investment policy, I examine in absolute terms instead of in relative terms because ω_i can be zero, making the ratios spike up and the graphs uninformative.

terms in the bracket describe the additional hedging demands: ω_i^u is the hedging requirement against the time-varying investment opportunity set, and ω_i^v is the hedge against the estimation risk. The three compositions, to some extent, correspond to the three factors that determine the value of $g_i(m)$, and their relative importance characterizes the state-dependent optimal policy.

Table 2 presents the magnitude and the relative weights of each composition for different m. As shown in Panel A, while the myopic component is the same regardless of the underlying information structure, its relative importance varies because the hedging components alter when the belief changes. For example, with no extra signal, $\frac{\omega_0^m}{\omega_0}$ is 90.259% and 101.330% for m equals to -0.05 and 1.5 respectively. At $m = r = 0.05$, one can see that $\omega_i^m = 0$ for $i = 0, 1$ and 2, while the two hedging components are non-zero (e.g. $\omega_1^u = 0.081$ and $\omega_1^v = -0.747$).

The aggregate hedging demand can be rewritten as

$$\omega_i^h \equiv \omega_i^u + \omega_i^v = \frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_i}}{\gamma\sigma_A^2}\frac{g_i'(m)}{g_i(m)} = -\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_i}}{\sigma_A^2}\frac{J_{iWm}}{WJ_{iWW}} = -\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_i}}{\sigma_A^2}\frac{\frac{\partial c_i}{\partial m}}{W\frac{\partial c_i}{\partial W}} \tag{28}$$

The common factor in the two hedging components, $\frac{g_i'(m)}{\gamma g(m)}$, reveals that the underlying purpose of the hedges is to hedge against unexpected changes in the investment environment that could lead to unfavorable drops in future payout as the risk-averse investors attempt to smooth consumption intertemporally. Such unexpected changes can arise from either shocks to the investment opportunity or mistakes in the estimation. I numerically show that (Panel E of Figure 1) $g(m) < 0$ for all $m \in \mathbb{R}$, and there exists a m^0 such that $g'(m^0) = 0$ and $g'(m)$ is negative (positive) to the left (right) of m^0.[22] Additional information assists not only through a direct improvement of estimation accuracy but also through an indirect reduction of the firm's vulnerability toward changes in the investment opportunity set. This makes the result different from previous literature.[23]

[22]Since $u(c) = (c^{1-\gamma} - 1)/(1-\gamma)$ and $\gamma > 1$, the utility level is negative. Therefore, $0 < g_i(m)/g_j(m) < 1$ if $g_j(m) > g_i(m)$. I will discuss the sign of $g_i'(m)$ in more detail in Section 3.3 when we discuss the implication on the investors' welfare.

[23]In line with the existing literature (Merton (1973), Brennan (1998), Barberies (2000), and Xia (2001)), the optimal investment policy includes an explicit term that captures the influence of estimation risk and another term that reflects the change in the variation in the investment opportunity set. However, in this paper, the two hedging components are interdependent. Information acquisition changes the estimation

More specifically, a negative (positive) $\rho_{\theta A}$ implies the shocks to the realized return are negatively (positively) correlated with the shocks to the expected return. Accordingly, the firm adjusts estimates upward (downward) after observing decreases in realized return and considers the risky asset as a proper (improper) hedge to variations in the investment opportunity. Therefore, ω_i^u is negative (positive) for $m < m^0$ and positive (negative) for $m > m^0$. Note that estimation risks make the firm more sensitive to changes in the opportunity set, which inevitably creates a stronger opportunity-uncertainty hedging demand, but at the same time dampens the risky asset's hedging role. Hence ω_i^u indeed encapsulates the direct effect of a time-varying investment environment and the indirect effect of information. In the absence of learning error, $\omega_0^u = \omega_1^u = \omega_2^u$ (perfect information), meaning in Panel B of Table 2, differences between ω_2^u and ω_0^u reflect the indirect influence of imperfect inference, and differences between ω_0^u and ω_1^u represent the implicit role of information on the firm's sensitivity to changes in the investment environment.

The learning-error hedging demand, ω_i^v, represents the direct incentive to hedge against imperfect inference, as the agent anticipates the persistence of learning error. Learning errors increase the likelihood of unfavourable shifts in payout and reduce the agent's confidence in the evaluation of the asset, which means the asset is riskier than it would be otherwise, and therefore less attractive. In order to hedge against unfavourable shifts in future consumption and minimize exposure to such error-related risk, a risk-averse agent has a lower intention to invest in the risky asset (long or short). Table 2 shows that the scale of ω_0^v is greater than the scale of ω_1^v, confirming that the direct hedging demand is weakened after information acquisition. One can also see that ω_i^v drops out if learning is perfect ($\overline{v}^{\mathcal{I}_2} = 0$). In sum, if $\sigma_{\theta A} + \overline{v_i} > 0$, the aggregate hedging demand is positive for $m \in (-\infty, m^0)$ and negative for $m \in (m^0, +\infty)$.[24] This is shown in Panel A of Figure 1.

Apart from the strategy to adopt (long or short) and the scale of investment, the investment policy also has another important dimension: the threshold to switch from long to short. When the investment opportunity set is constant, the agent adopts a short position when the expected rate of return is lower than r. In the presence of a time-varying investment opportunity set and imperfect learning, such a switching point, m_i^s, also exists and is

risk, which not only affects ω^v directly but also affects ω^u indirectly.

[24]The two hedging components reinforce (partially offset) each other when $\sigma_{\theta A} > (<) 0$.

given by

$$m_i^s + (\sigma_{\theta A} + \overline{v}^{\mathcal{I}_i})\frac{g_i'(m_i^s)}{g_i(m_i^s)} = r \Rightarrow m_i^s = r - (\sigma_{\theta A} + \overline{v}^{\mathcal{I}_i})\frac{g_i'(m_i^s)}{g_i(m_i^s)} = r - (\sigma_{\theta A} + \overline{v}^{\mathcal{I}_i})\frac{J_{iWm}}{J_{iWW}} \quad (29)$$

When the estimated return falls below m^s, the agent starts short selling the asset. The switching threshold shifts and is greater than r whenever $\sigma_{\theta A} + \overline{v}^{\mathcal{I}_i} > 0$ as $m^s > m^0$ and $g'(m^s) > 0$. As discussed previously, this occurs in two circumstances. The first one is when the two sources of risk reinforce each other ($\sigma_{\theta A} > 0, \overline{v}^{\mathcal{I}_i} > 0$, and ω_i^u and ω_i^v are of the same sign). Under such a circumstance, the risky asset is neither a good hedge to variations in the investment opportunity set nor to imperfect inference. The second one is when the estimation risk dominates ($\overline{v}^{\mathcal{I}_i} > -\sigma_{\theta A} > 0, \omega_i^u$ partially offsets ω_i^v). Although investment in the risky asset helps to hedge against unfavourable shifts in the investment environment, the estimation risk induced by holding the asset is too large, which makes the asset less attractive and induces the firm to switch earlier. In both scenarios, learning errors due to limited information reduce the willingness to hold the asset and increase the switching threshold. Information acquisition unambiguously reduces the threshold, expands the range where the firm is willing to hold the risky asset. This can be confirmed by column 5 ($m = r = 0.05$) of Panel E in Table 2, where $\omega_0(-0.775) < \omega_1(-0.666) < 0 < \omega_2(0.1)$.

Imperfect inference leads to inefficient investment policy – nonzero implicit and explicit hedging demand and higher switching threshold. Information acquisition largely reduces such inefficiency (see Panel B of Figure 1, where the magnitude of $\omega_2^h - \omega_1^h$ is smaller than the magnitude of $\omega_2^h - \omega_0^h$). The agent underinvests when the estimated rate of return from investing, either long or short, is not too small in absolute value. Envisioning the estimation risk, she acts conservatively by undertaking smaller long/short positions. On the contrary, the agent takes a bigger short position when the estimated return is relatively small in magnitude and close to the switching point m^s (i.e. $m^0 < m < m^s$), somehow creating an 'overinvestment' problem. Small m and the existence of estimation error means the likelihood of experiencing unexpected shifts in future consumption increases, which makes the holding of the asset riskier. Moreover, the agent is also uncertain about which strategy to adopt and when to switch. In this range, the estimation error, although stationary over states, can potentially lead to a significantly different strategy and outcome, which means the influence of estimation risk is more profound and the firm is more vulnerable to shocks in the investment opportunity. Therefore, hedging and learning are of first-order importance, which makes information imperative. The short position is scaled up because the embedded risk of longing the asset (asset volatility, estimation risk, investment opportunity set risk) is relatively large while the return from investment is relatively small. The absence of frictions such as transaction cost or shorting selling constraint means the firm can dynamically adjust the investment without any cost. Therefore, the firm still takes non-zero positions in the risky asset when $m^0 < m < m^s$ as uncertainties or risks incurred only scale investment rather than creating a wedge in the investment policy.

As m moves further away from this range, such scepticism decreases, and information plays a role mainly in guiding the amount of investment. In extreme cases, the myopic component dominates, and the value of information acquisition fades away. The above discussion leads to the corollary

Corollary 1 *The difference in the hedging demand generated by the information discrepancy is time-varying and converges to 0 as the estimate becomes extreme, that is, given any* $m \in \mathbb{R}$, $\lim_{m \to \pm\infty} \omega_1 - \omega_0 = 0$.

3.2 Payout policy

The optimal payout policies are given in Equation (20) and (23). Instead of maintaining a constant payout policy as in the standard Merton (1969, 1971), the firm's payout policy (the equityholders' consumption policy) is now time-varying and state-dependent, where $(1 - \gamma)^{-\frac{1}{\gamma}} g_i(m)^{-\frac{1}{\gamma}}$ is the marginal propensity to pay out. The payout yield and the comparison between different cases are plotted in Panel C and D of Figure 1.

Given the same level of total net worth, the payout yield increases as the firm's anticipated rate of return from investing increases, as shown in Panel C. All else equal, the payout also increases as the information set expands, which is shown in Panel D and gives

Corollary 2 *Given certain level of* W_t *and estimate* m_t*, the payout yield decreases with the estimation variance, and the ratios of payouts converge to 1 as the estimate becomes* *extreme, that is, for* $i, j \in \{0, 1, 2\}$ *and* $j > i$

$$\frac{c_j^*}{c_i^*} = \left(\frac{g_i(m)}{g_j(m)}\right)^{-\frac{1}{\gamma}} \geqslant 1 \quad \text{and} \quad \lim_{m \to \pm\infty} \frac{c_j^*}{c_i^*} = 1$$

Future payout is more expensive as the estimation error increases. With $\gamma > 1$, investors are unwilling to shift consumption across time because the intertemporal rate of substitution is low. Meanwhile, they value payout smoothing more than the less risk averse investors. Higher estimation error means that the investors need to sacrifice more for the same consumption tomorrow, i.e. the opportunity cost of current consumption increases. Intertemporal substitution becomes expensive, and the cost of achieving intertemporal smoothing increases. Consequently, the current payout shrinks due to the greater uncertainty in the future payout. In the numerical analysis, I also show that the greater the estimation risk, the higher degree of payout smoothing.[25] Therefore, payout smoothing emerges as a consequence of risk aversion as well as the persistence of incomplete information and imperfect learning.

On the one hand, information gathering alleviates investment inefficiency and reduces required hedging demand and learning effort. On the other hand, collecting more information reduces the need to preserve wealth to buffer unanticipated shocks in the future. Therefore, the improvement in investment efficiency and reduction in precautionary saving increases disposable income. Moreover, increasing confidence in the assessment of the future investment environment encourages the firm to pay out more and makes intertemporal substitution less expensive. Overall, one could expect a positive influence of information acquisition on the payout policy because of a better understanding of future outlook, which is verified in Panel D in Figure 1 (all the ratios, c_i/c_j with $i > j$, are above one).

Panel D in Figure 1 shows a bimodal shape of payout ratios, and the two modes straddle on the two sides of m^0. Recall that m^0 is the point where the value of the risky investment opportunity reaches the minimum. The positive influence of information acquisition shrinks for anticipated rates of return that are close to m^0 as the investment opportunity is less valuable. As managerial beliefs move away from this point, the risky asset becomes more

[25] With the parameter values specified in Table 1, for a given level of W_t, the variances of the payout yield c_i/W are 52.54, 52.63, and 53.06 for the 0-signal, 1-signal and perfect information cases respectively.

valuable in improving payout and welfare. When the estimated rate of return from investing is still relatively low, especially around the switching threshold, estimation risk has a more significant impact, making learning more important. Therefore, the positive influence of information acquisition becomes more salient. In these regions, the marginal utility of consumption remains high due to the relatively low level of payout. As the perceived return gravitates toward extremes, learning becomes less valuable, and so is information. Therefore, the information-induced consumption converges to 0 as $m \to \pm\infty$.

3.3 Welfare implications

The investors want to maximize the lifetime utility from the payout. Learning based on different information sets alters the payout pattern and therefore influences the welfare the investors could obtain from investing in the firm. Information acquisition affects the payout and welfare through $g_i(m)$, as shown in (18). As discussed at the beginning of Section 3, three factors together determine the value of $g_i(m)$: the investment, hedging and information. Learning accuracy affects the magnitude and intensity of these factors. Panel E and F of Figure 1 plot $g_i(m)$ and the differences as functions of m, respectively.

Since the firms can take either long or short position, it means that the downside occurs when the $|m|$ is relatively small while the upside refers to cases where $|m|$ is large. Panel E shows that, all else equal, the investment opportunity becomes more valuable when the estimated rate of return from investing (both long and short) increases; in other words, where the investment factor dominates. In good times, the agent expects the firm's total net worth to grow at a higher rate, and consequently is willing to increase payout, which leads to higher consumption and welfare. The valuation reaches the minimum at m^0 where $g'(m^0) = 0$. At this point, the aggregate hedging component, as shown in Section 3.1, is zero, shutting down the hedging contribution of the risky asset. The investment role is also tiny because the anticipated rate of return from investment is small, leaving the learning roles. As a result, the benefits from investing in the risky asset reduce, and the risky asset becomes less valuable. Substituting $g'(m^0) = 0$ into (21) and (24) shows

$$\underbrace{\left(r(1-\gamma) - \rho + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{(m-r)^2}{\sigma_A^2}\right)g_i(m) + \frac{\gamma}{1-\gamma}(1-\gamma)^{-\frac{1}{\gamma}}g_i(m)^{-\frac{1}{\gamma}}}_{\text{investment factor}} + \underbrace{\frac{1}{2}\Psi_i g_i''(m)}_{\text{information factor}} = 0 \tag{30}$$

where Ψ_i denotes the reduction in estimation error from $\mathcal{I}_i$. The above equation shows that only the investment factor and information factor remain. Therefore, the discrepancy in the welfare around this point mainly reflects the difference in learning based on differentiated information sets. As m deviates from this point, the agent starts to rely on the asset to hedge against uncertainty in the investment opportunity, and the indirect effect of information also starts to emerge, which reinforces the positive influence of information acquisition and generates a bimodal shape shown in Panel F.

Also, the firm values the investment opportunity more when it has a better understanding of it. The investors benefit from information acquisition as a higher assessment accuracy of the investment environment induces more efficient investment decisions and a higher payout. The positive effect is particularly strong in bad times where the expected rate of return is relatively small. The welfare improvement fades out as the anticipated rate of return gravitates toward extremes. As the instantaneous return from investment becomes relatively large, the investment factor dominates the other two factors, and the agent becomes more myopic in setting the policies. Payout is relatively high, and the marginal utility of consumption for the investors is relatively low. Hence, the estimation accuracy is no longer the primary concern, and the incremental welfare generated by information gathering becomes negligible. The discussion leads to

Corollary 3 *Given any total net worth $W_t > 0$ and an estimation $m_t \in \mathbb{R}$, the indirect utility increases as the learning error decreases. The learning-error induced differences converge to zero as the estimated rate of return becomes extreme, that is*

$$J_2(W,m) \geq J_1(W,m) \geq J_0(W,m) \quad \textit{and} \quad \lim_{m\to\pm\infty}\frac{J_1^*(m,W)}{J_0^*(m,W)} = 1 \tag{31}$$

4 The Cost and Value of Information Acquisition

So far, I have assumed that information acquisition is costless.[26] Advances in modern information technologies have reduced the cost of information gathering significantly. Regulators also have been adapting to the unprecedented evolution in data production. For example, the SEC introduced the EDGARD system to change the paper-based corporate disclosure to a digital one in 1993 and allowed firms to use social media platforms such as Facebook and Twitter to release essential messages in 2013. Such progress enables timely information dissemination and reduced acquisition cost significantly (Gao and Huang (2018), Goldstein, Yang and Zuo (2020)). Meanwhile, casual observation reveals that agents nowadays do pay for information or services in order to facilitate better decision making. Curran (2019) forecasts the total revenue of the U.S. financial data service providers to reach \$17 billion in the year 2024 from \$15.4 billion in 2019. Active equity funds, sell-side analyst reports, investment advisors etc. have been playing important roles in directing capital flows. Thus, addressing the value and cost of information acquisition becomes economically crucial as it helps to understand the role of growing information intermediaries (e.g. mutual funds, wealth managers, consulting firms, data providers) and the value they create. It also sheds light on how costs affect the attractiveness of information services and reconciles empirical evidence.

4.1 Certainty equivalent of wealth

The previous discussion indicates that the extra signal's influence is not stationary even though its contribution to learning is stable ($\overline{v}^{\mathcal{I}_0} - \overline{v}^{\mathcal{I}_1}$ is constant), suggesting that the willingness to pay for information acquisition varies over time.

To examine the variation of information value, two measures are used. The first one is the certainty equivalent of wealth (CEW), with which the agent is indifferent between keeping the default information structure and conducting extra information search. Mathematically, it is defined as $\delta_{ij}W$ such that $J_i(W(1+\delta_{ij}), m) = J_j(W, m)$, with $i, j \in \{0, 1, 2\}$. For

[26] Cost can be of a pecuniary form (e.g. management fee, cost to the brokers) or a nonpecuniary form (e.g. time and effort).

example, $\delta_{01}(m)W$ is the amount of wealth the agent with $\mathcal{I}_0$ is willing to relinquish for the extra signal that expands the information set to $\mathcal{I}_1$. $\delta_{ij}(m)$ can be written as

$$\delta_{ij}(m) = \left(\frac{g_j(m)}{g_i(m)}\right)^{\frac{1}{1-\gamma}} - 1 = \left(\frac{c_i}{c_j}\right)^{-\frac{\gamma}{1-\gamma}} - 1 \tag{32}$$

This indicates that the main driver of CEW is the variation in payout. All else equal, the CEW is increasing in the degree of risk aversion, meaning a more risk-averse investor values the signal more. Panel A in Figure 4 plots $\delta_{ij}(m)$, and the results are consistent with the discussion in Sections 3.2 and 3.3. First, it shows that, despite the time-fluctuating influence of the signal, there is always a positive value attached as long as perfect learning can not be achieved, and the value can be quite significant in some states. Second, the certainty equivalent of wealth for information is also bimodal in m, and the two modes lie on the two sides of m^0. The signal is worth less for estimates around m^0 because the benefits from investing are relatively low. Accordingly, the learning effort dampens, resulting in lower demand for information. As beliefs move away from m^0, the investment opportunity becomes more valuable for its roles in hedging, investment and learning. When the anticipated rate of return is still not large enough, especially around the switching threshold, the estimation risk imposes a greater threat to the firm's decision-making, the asset is more 'risky', and the value of information surges. The agent has a lower intention to acquire new signals when she holds a high expectation about the investment return. Thus, the corollary follows

Corollary 4 *The certainty equivalent of wealth (CEW) of information is time-varying, with*

$$\lim_{m \to \pm\infty} \delta_{ij}(m) = 0 \quad \textit{for} \quad i, j \in \{0, 1, 2\} \quad \textit{and} \quad i < j \tag{33}$$

4.2 Cost distortion effect

Assume the cost of information acquisition is proportional to the total amount of investment, and the intertemporal budget constraint becomes

$$dW_t = dA_t - \alpha A_t dt + dD_t - cdt = [(r + (m_t^{\mathcal{I}_i} - r - \alpha)\omega_t)W_t - c_t]dt + \omega_t \sigma_A W_t d\hat{Z}_{At}^{\mathcal{I}_i} \tag{34}$$

where $\alpha A_t dt$ represents the instantaneous proportional cost. Another way to interpret the cost is that the agent has to forgo α rate of return in order to obtain the signal. Such proportional cost structure is common in finance; for example, the mutual fund fee, the management and incentive fee in private equity funds.[27] Now the agent has to balance the costs and benefits of information gathering. With the cost consideration, the agent's evaluation of the risky investment opportunity changes. The proportional cost introduces the following extra terms to Equation (24):

$$\frac{\alpha(1-\gamma)}{2\gamma\sigma_A^2}\Big[(\alpha - 2(m-r))g_1(m) - (\sigma_{\theta A} + \overline{v}^{\mathcal{I}_1})g_1'(m)\Big] \tag{35}$$

A direct cost consequence is a reduction in return from investment as the firm can only collect a fraction of it, which is reflected by the first term in the bracket. However, the new signal also reduces the riskiness and uncertainty, which offsets some of the negative cost impacts. This is shown by the second term in the bracket. If acquiring an extra signal incurs a cost that overtakes the benefit it brings, the optimal decisions are subsequently distorted.

To examine the distortion effect, I compare the optimal policies under costly acquisition, with those under costless acquisition, for a given W_t. Panels A and B of Figure 2 plot the differences between $\{(\omega_1^\star(m), c_1^\star(m))|\alpha \neq 0)\}$ and $\{(\omega_1^\star(m), c_1^\star(m)|\alpha = 0\}$.[28] Unsurprisingly, the cost induces the agent to shrink the investment scale as the realized return is lower while the risk involved remains unchanged. Because of the reduction in the anticipated return and distortion in the investment, payout is revised downward accordingly. As a result, costs unambiguously make the investors worse off, and the detrimental effect increases as costs rise.

Panels C and D of Figure 2 compare the optimal policies between the costly 1-signal case

[27]This is equivalent to subtracting the instantaneous fee ratio α from the expected rate of return, which means the return process of the risky asset becomes $dA_t/A_t = (m^{\mathcal{I}_1} - \alpha)dt + \sigma_A d\hat{Z}_{At}^{\mathcal{I}_1}$. Dangl, Wu and Zechner (2006) also model the mutual fund management fee using a similar setup in a continuous-time model to examine the interaction of mutual fund managers' turnover, performance, and risk-taking, and investors' learning and capital flow.

[28]In the numerical result, I only focus on the side where a long position is taken with $\alpha = 0$ and discuss the cost impacts when α is set to be 0.0015, 0.005 and 0.01. For the other side, the analysis can be applied accordingly.

and the 0-signal case. With low cost (dotted lines, $\alpha = 0.0015$), the benefit dominates. While investment and payout are shrunk accordingly, the investor is still better off with the signal. As the cost increases, the distortion effect emerges. With slightly higher cost (dashed lines, $\alpha = 0.005$), the newly acquired signal mitigates the inefficient investment problem compared to the 0-signal case, but only in states where learning accuracy is imperative. As can be seen from Panel C, the improvement effect of the costly signal is particularly strong when the estimates are relatively small and close to the switching threshold. Estimation error leads to a disparity between the realized and the estimated return and results in inefficiency. The firm is more sensitive to the estimation risk in this range and, thus, demands higher assessment precision. Information is deemed to be exceptionally important here as the agent may mistakenly adopt a completely opposite strategy due to a small estimation error. With this premise, the information acquisition, although costly, mitigates inefficiency and gives rise to a higher payout compared to the 0-signal case.

However, as beliefs gravitate toward extreme (when $m > 0.9985$ for $\alpha = 0.005$), the demand for better inference decreases and the cost incurred dominates. The agent has to scale back investment and reduce the payout to levels that are even lower than under the 0-signal case in order to cover the cost. In this case, the costly acquisition aggravates inefficiency. This can also occur when the cost of information acquisition is too high relative to its embedded informativeness (thin solid line, $\alpha = 0.001$), in which distortion happens even when the demand for better learning is high.

Figure 3 presents the ratios of the welfare, $\frac{g_1(m;\alpha)}{g_0(m)}$, for different levels of costs and a given level of W_t. For small α (0.0015), the investors' welfare, although slightly lower than the zero cost case, improves compared to the 0-signal cases.[29] For medium α (0.005), the extra signal does not always improve indirect utility. The cost overtakes the estimation risk concern when $m > 0.9985$, hampering the investors' welfare. For large α (0.01), the cost erodes the benefits and the firm finds it too expensive to conduct costly information search.

Overall, we can see that the cost distortion effect is state-dependent. The result implies that the value of information is countercyclical – it is high (low) in the downside (upside) where the expected rate of return from investing is low (high). Moreover, it also depends on

[29]Note that $g_i(m) < 0$; therefore, if a signal improves welfare, $\frac{g_1(m;\alpha)}{g_0(m)} < 1$.

the agent's learning path. This means that a positive signal leading to investor optimism (large m) can crowd out the subsequent motivation for information collection. A negative signal that revises the beliefs downward can induce a greater effort to obtain information. These results generate a variety of empirical implications that will be discussed in Section 5.

4.3 The cutoff cost boundary

In this section, I consider a second measure of information value – the cutoff cost boundary. Denote α_{01} the cutoff cost, i.e. the maximum percentage cost the agent is willing to pay to expand her information to $\mathcal{I}_1$ from $\mathcal{I}_0$. Signals reducing the estimation error from $\overline{v}^{\mathcal{I}_0}$ to $\overline{v}^{\mathcal{I}_1}$ are deemed to be too expensive if they cost more than α_{01}. In this case, the agent prefers a poorer assessment over an expensive one. I show that the critical α_{01} is the solution to the following[30]

$$\begin{aligned}
&[(r - m)g_0(m) - (\sigma_{\theta A} + \overline{v}^{\mathcal{I}_1})g_0'(m)]\alpha_{01} + \frac{1}{2}g_0(m)\alpha_{01}^2 \\
&= (m - r)(\overline{v}^{\mathcal{I}_0} - \overline{v}^{\mathcal{I}_1})g_0'(m) + \frac{1}{2}\Big[2\sigma_{\theta A}(\overline{v}^{\mathcal{I}_0} - \overline{v}^{\mathcal{I}_1}) + (\overline{v}^{\mathcal{I}_0})^2 - (\overline{v}^{\mathcal{I}_1})^2\Big]\frac{g_0'(m)^2}{g_0(m)} \qquad (36)\\
&+ \frac{1}{2}\sigma_A^2\frac{\gamma}{1-\gamma}\Big[\Big(\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_0}}{\sigma_A}\Big)^2 - \Big(\xi_A + \xi_S + 2\rho_{AS}\xi_A\xi_S\Big)\Big]g_0''(m)
\end{aligned}$$

By solving the quadratic function, α_{01} can be uniquely pinned down for each m and the expression is given by (57) in the Appendix.[31] The boundary is obtained when the marginal cost equals the marginal benefits. The left-hand side of (36) represents the cost of the signal: the firm can only collect a fraction of return, and the total net worth is now accumulated at a lower rate. The right-hand side captures the benefit: estimation risk is mitigated, and the firm is less sensitive to variation in the opportunity set. The firm is more robust to changes in the investment environment and therefore requires lower hedge and precautionary saving.

[30] α_{ij} denotes the maximum rate of return the agent with $\mathcal{I}_i$ is willing to relinquish in exchange for information $\mathcal{I}_j$. I defer the tedious derivation to the Appendix.

[31] The quadratic shape comes from the symmetric of the g_i function as no restrictions are imposed on the investment strategy. The agent can switch from long to short when the expected rate of return is below the switching threshold. Solving the quadratic function leads to two distinct roots for each m (one positive and one negative). The positive (negative) solution is selected for m such that a long (short) position is taken. Otherwise, the cost becomes a stream of income rather than expenses.

Panel B in Figure 4 plots α_{ij} in m and shows results that are in line with the previous discussion. At each state, the agent is willing to trade off a certain degree of expected return for a better assessment; however, such willingness varies. The cutoff cost increases as the informativeness increases. The agent is more willing to pay when uncertainty lies not only in the amount of investment but also in the sign the position. The critical α is also high in bad times, where the firm relies on the investment opportunity for hedging and learning, yet the return from investment is low. The maximum cost the firm is willing to pay drops as the anticipated return from investment increases. The high estimated rate of return reduces the influence of estimation risks, making the investment more attractive and the hedging demands being dominated. Also, the absolute amount of return relinquished to information providers jumps. Consequently, the maximum cost that the agent is willing to absorb descends. The time-varying cutoff cost suggests that a constant cost structure can distort the firm's policies in some states, even though the quality of the signal remains unchanged. The value of information and thus the motivation for information gathering, ceteris paribus, depends not only on the cost, but also the agent's beliefs and her learning path.

5 Empirical Implications and Applications

The general model presented has applications in various aspects of financial economics and provides empirical implications that are relevant to a variety of literature. In this section, I demonstrate the model's implications for three cases of application: passive and active investing, public information and private information production, and investment and payout.

5.1 Passive and active investment

The first implication of the model is that the value of information acquisition is time-varying and depends on the cost incurred. Without additional information, the agent adjusts her strategy passively in response to new observations of realized return. With active information collection, the agent uses the newly acquired signal S_t to guide the investment and payout decisions. Such a setup allows us to map the generic model to the discussion of active and passive investment and draw implications. Within the content of this paper, the extra signal can be thought of as the service provided by financial intermediaries such as mutual funds. The investors compare the passive investment (0-signal) with the active investment (1-signal).[32]

Established research in individual investors and mutual funds shows that active investors, who presumably make more informed choices, underperform their passive counterparts, net of fee and transaction cost.[33] This model demonstrates that, all else equal, active investment creates information advantage and 'outperforms' the passive investment before cost.[34] Active information gathering enables better evaluation of the investment opportunity set, thereby improving decision making. However, the benefits created by active management can be overwhelmed by the cost incurred, in particular when learning is less important and when the cost is high. Cost overturns the 'outperformance' and induces inefficient decision-making.

Table 3 presents the cutoff cost, α_{ij}, calculated by the model and the corresponding changes in the indirect utility function, $\frac{g_i(m)}{g_j(m)} - 1$.[35] With the baseline parameter values specified in Table 1, for a signal that reduces the estimation variance by 16%, the maximum cost a risk averse investor is willing to bear (α_{01}) is 0.7% of the total investment when the expected rate of return is 12% (7% net of the risk-free rate). This is much lower than the conventional 1 − 2% expense ratio incurred when investing in active mutual funds. For a 40% anticipated rate of return, $\alpha_{01} = 1\%$. In comparison, investment in private equity funds often incurs a 2% management fee plus an extra 20% performance charge.

[32]Dangl, Wu and Zechner (2006) use a continuous-time learning model to examine a mutual manager's career, performance and risk-taking behavior, where investors learn about the managers' stock-picking ability by observing the fund performance. In their model, they assume competitive provision of capital and the management company try to maximize the value of the whole company. Different from this paper, my paper focuses on investors' decisions, where they compare between passive and active investment options.

[33]See Jensen (1968), Wermers (2000), Barber and Odean (2000) and others.

[34]Note that in this paper, the performance of an investment is measured in terms of the discrepancy between the financial decisions under incomplete information and the ones under perfect information (i.e. the efficiency of the investment policy, and differences in the payout and the degree of improvement in the investors' welfare) rather than abnormal return as in the traditional asset pricing literature.

[35]Note that $g_i(m)$ is negative for all m. Therefore, an improvment in the indirect utility is represented by a negative $g_i(m)/g_j(m) - 1$.

The cutoff boundary increases in the informativeness of the signal. If there exists a signal that can provide perfect information, the investor is willing to pay $\alpha_{02} = 6\%$ of the total investment when $m = 40\%$, with which the welfare is raised by 34.7%. Costs exceeding the cutoff boundary exhaust the benefits created, induce 'underperformance' and dampen the investors' interest in information services.

Despite the persistent 'underperformance' of costly active funds, the active management industry remains gigantic, which is puzzling given the widely available less expensive passive alternatives.[36] Moskowitz (2000) proposes a hypothesis stating that investors are willing to tolerate underperformance for the value the active funds can create when they are needed the most; for example, in a recession. Some papers have already documented outperformance of active funds in recessions (Glode (2011), Kacperczyk, Van Nieuwerburgh, and Veldkamp (2014, 2016), and others). Glode (2011) proposes a model that is in line with the Moskowitz (2000) hypothesis. In his model, the manager exerts greater effort and delivers a higher return in states where the marginal value of consumption and willingness to pay for the service of investors are high. Empirically, he finds that underperforming funds that charge high fees can deliver countercyclical abnormal returns.

Consistent with these observations and hypothesis, my model shows a time-varying value of active investing. Information service is more critical for an risk-averse investor when the investment environment is more volatile,[37] and in bad times where the expected return from investment is relatively low. Under these circumstances, risks from investment, including the return volatility, the estimation risk and the opportunity uncertainty risk, are too high relative to the estimated return. Payout is relatively low and therefore the investor's marginal utility of consumption is high, which leads to a surge in the motivation to invest in costly active management to improve decision making. This suggests that investors are willing to accept a certain degree of cost distortion in some states in exchange for better decision-making in some other states where consumption or payout is needed the most. Put differently in the context of capital flow, the model implies more capital inflows to the active

[36]'US money market funds waive fees to stave off negative returns,' *Financial Times*, 29 May 2020; 'Fidelity's zero-fee campaign spurs $6.6bn of inflows,' *Financial Times*, 26 November 2018.

[37]In an unreported result (available upon request), I show that increases in the volatility of the investment opportunity set make information acquisition more valuable.

management industry in bad times than in good times.

In the recent pandemic turmoil, performance data show that the UK active equity funds are outperforming their passive index rivals.[38] My model suggests a state-dependent outcome for the race between active and passive investment in terms of improving risk averse investors' welfare. Investigating the time-series of the relative performance between active and passive investors and how it directs the capital flow can improve understanding of the active management industry.

5.2 Public information and private information production

Another implication of the model is that the demand for additional information aid depends on the informativeness of the endowed information set as well as the agent's learning curve. On the one hand, the model shows that information acquisition has a zero value if the realized return is a sufficient statistic, which means that informative public signals crowd out private information production. Whether or not public information crowds out managers' or investors' private information production has been discussed in the corporate disclosure literature. For review analysis, I refer to Goldstein and Yang (2017).[39]

On the other hand, this paper shows that the demand for private signals is a function of the agent's belief, which is a path-dependent process that evolves in response to new observations of the signals in her information set. The valuation, hence the motivation, of information acquisition decreases when the agent's belief about the expected return conditional on information up to time t is relatively high. This means that if a public signal, for example the past return, revises the agent's prediction upwards, even though it is a noisy signal, it could potentially crowd out private information production. On the contrary, if the agent's beliefs drop to the range where estimation risk is a first-order concern after

[38] 'Active funds have the edge on passives in the week of turmoil,' *Financial Times*, 13 March 2020.

[39] Gao and Huang (2018) and Goldstein, Yang and Zuo (2020) study how modern information dissemination technology affects outsiders' information production by looking at the EDGAR system. Gao and Huang (2018) show that after the full implementation of the EDGAR system, trades by investors with internet access become more informative, and the quality of sell-side analyst reports increases. They suggest that public information encourages private information production (crowd-in effect). Goldstein, Yang and Zuo (2020) examine the staggered implementation process of the EDGAR system and find a decrease in the investment-to-price sensitivity. They suggest that the propagation of public information can crowd out private information acquisition.

observing a public signal, the demand to search for private information surges. In this case, public signals encourage private information production (crowd-in effect). Therefore, the model suggests the coexistence of the 'crowd-in' and 'crowd-out' effect. Furthermore, the model predicts countercyclical information production: the intention of private information production increases in bad times where learning is essential and decreases in good times.

Countercyclical private information production also implies a time-varying power of information (for example, corporate announcements and media) in explaining the investors' investment, consumption and saving decisions. The response of investors to the same piece of information differs across states. Bhattacharya et al. (2009) show that, although there exists a positive relation between media coverage and internet IPOs in the 1990s, the media frenzy fails to explain the internet bubble. The media can only explain 2.9% of the massive 1646% return difference between the internet and non-internet firms. This paper provides theoretical support for the lack of explanatory power of information in extreme circumstances. Examining how investors' sentiment or belief influence the interaction between public information and private information production could be potentially inspiring and provide insights for policymakers.

5.3 Investment and payout

The model also sheds light on the real effect of information and learning. In particular, it suggests that information improves investment efficiency and mitigates underinvestment by enhancing the agent's understanding of the risky investment opportunity set. The implication is consistent with findings in the empirical literature that studies how information acquisition affects investment decisions for firms, mutual funds and venture capitals (Coval and Moskowitz (1999, 2001), Chen et al. (2010), Giroud (2013), and others). These papers find that financial entities with information advantage are less bounded by estimation risks and are able to make more efficient investment choices.

Estimation risk also induces the firm to decrease payout and smooth payout more compared to the circumstance where information is perfect. Better information improves the firm's confidence in the valuation of future return, which greatly reduces the hedging and precautionary demands. As a result, payout is higher and more responsive to income shocks, i.e. less smooth. This also means changes in payout and investment policy reflect the firm's learning path and confidence in future evaluaton. All these results suggest measures or policies that reduce information collection barriers or costs could potentially mitigate underinvestment and improve efficiency and welfare.[40]

6 Conclusion

The paper presents a generic model to understand the motivation and influence of information acquisition. In particular, I study how the decision to acquire an extra signal affects learning paths and the dynamics of investment and payout for a firm when information is incomplete, and learning is imperfect. Improved learning through information acquisition alleviates investment inefficiency, encourages higher payout and enhances the investors' welfare. However, the benefits can be overwhelmed by the cost incurred, distorting the agent's motivation to search for information. The benefits dominate when accurate inference is more important. Thus, the value of information acquisition is state-dependent. Information is more valuable when the investment environment is more volatile, when the firm is more vulnerable to changes in the investment environment, and when the investors' marginal utility of payout is higher.

The model offers plausible explanations for otherwise puzzling observations; for example, why investors invest in active funds despite the persistent underperformance after cost and the widely available low-cost passive alternatives. It suggests that the ability to deliver positive outcomes in states where information and consumption are needed the most (e.g. in a recession) justifies the reason to engage in active information search. Additionally, the model predicts a countercyclical production of private information, and sheds light on the real effect of information and learning on firms' investment and payout. It also has implications for the time-series and cross-sectional variation in market reactions to

[40]Giroud (2003) shows that the opening of new airline routes increases the headquarter's plant investment by 8% − 9% and attribute the expansion of investment to reductions in the information gap. Chen et al. (2020) use the introduction of the high-speed rail in China as an exogenous shock and find that reduction in information acquisition cost results in greater information production, more accurate analyst forecasts and better analyst recommendations.

information, and suggests a state-dependent value of financial information intermediaries and a time-varying explanatory power of information on financial behavior.

Admittedly, the model is agnostic about the supply side of information and the agent's capacity to process information. Future research could incorporate the supply-side limit and demand-side constraint. For example, if investors can only choose signals from a limited pool, how do the choices of information acquisition depend on their endowed information set and information pool, and how do information providers tailor the supply to acquirers' preferences? The corresponding welfare implications require an equilibrium framework in future research. Finally, I abstract from many other market frictions, such as transaction costs and short-selling constraints. It would be interesting to see how these frictions affect the main results and what kind of empirical implications they generate about information acquisition behavior.

Appendix

Proof of Proposition 1

The continuous-time Bayesian inference process relies on non-linear filtering theory (see Lipster and Shiryaev (2001a, 2001b)). The prior belief is assumed to be Gaussian with mean m_0, and variance v_0. Thus, conditional of the information set, the updating rules for estimation $m_t^{\mathcal{I}_i} = \mathbb{E}(\theta_t|\mathcal{F}_t^{\mathcal{I}_i})$ and the variance $v_t^{\mathcal{I}_i} = \mathbb{E}((\theta_t - m_t)^2|\mathcal{F}_t^{\mathcal{I}_i})$ are unique continous $\mathcal{F}_t^{\mathcal{I}_i}$-measurable solutions to the following

$$dm_t^{\mathcal{I}_i} = (a_0 + a_1 m_t^{\mathcal{I}_i})dt + (\sigma_\theta \circ \sigma^{\mathcal{I}_i} + v^{\mathcal{I}_i} B_1^T)(\sigma^{\mathcal{I}_i} \circ \sigma^{\mathcal{I}_i})^{-1}(d\xi_t^{\mathcal{I}_i} - \mu_1 m_t^{\mathcal{I}_i} dt) \tag{37}$$

$$dv_t^{\mathcal{I}_i} = 2a_1 v_t^{\mathcal{I}_i} + (\sigma_\theta \circ \sigma_\theta) + (\sigma_\theta \circ \sigma^{\mathcal{I}_i} + v^{\mathcal{I}_i} B_1^T)(\sigma^{\mathcal{I}_i} \circ \sigma^{\mathcal{I}_i})^{-1}(\sigma_\theta \circ \sigma^{\mathcal{I}_i} + v^{\mathcal{I}_i} B_1^T)^T \tag{38}$$

where

$$\sigma^{\mathcal{I}_0} = \sigma_A \quad \text{and} \quad \sigma^{\mathcal{I}_1} = \begin{bmatrix} \sigma_A \\ \sigma_S \end{bmatrix}; \qquad B_1^{\mathcal{I}_0} = 1 \quad \text{and} \quad B_1^{\mathcal{I}_1} = \begin{bmatrix} 1 \\ b_1 \end{bmatrix}$$

$$\sigma_\theta \circ \sigma^{\mathcal{I}_0} = \sigma_{\theta A} \quad \text{and} \quad \sigma_\theta \circ \sigma^{\mathcal{I}_1} = \begin{bmatrix} \sigma_{\theta A} & \sigma_{\theta S} \end{bmatrix};$$

$$\sigma^{\mathcal{I}_0} \circ \sigma^{\mathcal{I}_0} = \sigma_A^2 \quad \text{and} \quad \sigma^{\mathcal{I}_1} \circ \sigma^{\mathcal{I}_1} = \begin{bmatrix} \sigma_A^2 & \sigma_{AS} \\ \sigma_{AS} & \sigma_S^2 \end{bmatrix}; \qquad d\xi_t^{\mathcal{I}_0} = \frac{dA_t}{A_t} \quad \text{and} \quad d\xi_t^{\mathcal{I}_1} = \begin{bmatrix} \frac{dA_t}{A_t} \\ dS_t \end{bmatrix}$$

That is, $\sigma_\theta \circ \sigma^{\mathcal{I}_i}$ is the variance-covariance matrix of unknown drift and observations, $\sigma^{\mathcal{I}_i} \circ \sigma^{\mathcal{I}_i}$ is the variance-covariance matrix of the observations, $d\xi_t^{\mathcal{I}_i}$ is the "surprise" from new observation. Simplifying the expression leads to the corresponding updating

$$dm_t^{\mathcal{I}_0} = (a_0 + a_1 m_t^{\mathcal{I}_0})dt + \frac{\sigma_{\theta A} + v_t^{\mathcal{I}_0}}{\sigma_A^2} d\hat{Z}_{At}^{\mathcal{I}_0} \tag{39}$$

$$dm_t^{\mathcal{I}_1} = (a_0 + a_1 m_t^{\mathcal{I}_1})dt + \zeta_{At} d\hat{Z}_{At}^{\mathcal{I}_1} + \zeta_{St} d\hat{Z}_{St} \quad \text{with} \tag{40}$$

$$\zeta_{At} \equiv \frac{\zeta_{1t} - \rho_{AS}\zeta_{2t}}{1 - \rho_{AS}^2} \quad \text{and} \quad \zeta_{St} \equiv \frac{\zeta_{2t} - \rho_{AS}\zeta_{1t}}{1 - \rho_{AS}^2} \tag{41}$$

$$\zeta_{1t} \equiv \frac{v_t^{\mathcal{I}_1} + \sigma_{\theta A}}{\sigma_A} \quad \text{and} \quad \zeta_{2t} = \frac{b_1 v_t^{\mathcal{I}_1} + \sigma_{\theta S}}{\sigma_S} \tag{42}$$

The evolutions of the estimation variances are given in (10) and (11). The stationary variances are the solutions to (10) and (11) and are given in (43) and (44). Substituting the stationary variances yields (6) and (7).

Proof of Proposition 2

The stationary variances are the solutions to Equations (10) and (11). Solving the two quadratic equations and selecting the positive roots leads to

$$\overline{v}^{\mathcal{I}_0} = a_1\sigma_A^2 - \sigma_{\theta A} + \sigma_A^2 \sqrt{a_1^2 - 2\rho_{\theta A} a_1 \frac{\sigma_\theta}{\sigma_A} + \frac{\sigma_\theta^2}{\sigma_A^2}} \tag{43}$$

$$\begin{aligned} \overline{v}^{\mathcal{I}_1} = \frac{\sigma_A\sigma_S}{b_1^2\sigma_A^2 - 2b_1\rho_{AS}\sigma_A\sigma_S + \sigma_S^2} \Bigg[& a_1(1 - \rho_{AS}^2)\sigma_A\sigma_S + b_1\rho_{AS}\sigma_A\sigma_\theta(\rho_{\theta A}\rho_{AS} - \rho_{\theta S}) - \rho_{\theta A}\sigma_S\sigma_\theta \\ & + \rho_{AS}\rho_{\theta S}\sigma_\theta\sigma_S + (1 - \rho_{AS}^2)\sigma_A\sigma_S \sqrt{\frac{H^{\mathcal{I}_1}}{(1 - \rho_{AS}^2)\sigma_A^2\sigma_S^2}} \Bigg] \text{, where} \end{aligned} \tag{44}$$

$$\begin{aligned} H^{\mathcal{I}_1} \equiv\ & a_1^2(1 - \rho_{AS}^2)\sigma_A^2\sigma_S^2 + 2a_1\sigma_A\sigma_S\sigma_\theta\Big(b_1\sigma_A(\rho_{AS}\rho_{\theta A} - \rho_{\theta S}) + \sigma_S(\rho_{\theta S}\rho_{AS} - \rho_{\theta A})\Big) \\ & + \sigma_\theta^2\Big(b_1^2(1 - \rho_{\theta A}^2)\sigma_A^2 + 2b_1\sigma_A\sigma_S(\rho_{\theta A}\rho_{\theta S} - \rho_{AS}) + \sigma_S^2(1 - \rho_{\theta S}^2)\Big) \end{aligned}$$

By setting $a_1 = 0$, the stationary variances for the 0-signal and 1-signal cases are

$$\overline{v}^{\mathcal{I}_0} = -\sigma_{\theta A} + \sigma_A^2\sqrt{\frac{\sigma_\theta^2}{\sigma_A^2}} = \sigma_\theta\sigma_A(1 - \rho_{\theta A}) \tag{45}$$

$$\overline{v}^{\mathcal{I}_0} = \frac{\sigma_A\sigma_S}{b_1^2\sigma_A^2 - 2b_1\rho_{AS}\sigma_A\sigma_S + \sigma_S^2}\Bigg[b_1\rho_{AS}\sigma_A\sigma_\theta(\rho_{\theta A}\rho_{AS} - \rho_{\theta S}) + \sigma_S\sigma_\theta(\rho_{AS}\rho_{\theta S} - \rho_{\theta A}) \tag{46}$$

$$+ (1 - \rho_{AS}^2)\sigma_A\sigma_S\sigma_\theta\sqrt{\frac{b_1^2(1 - \rho_{\theta A}^2)\sigma_A^2 + 2b_1\sigma_A\sigma_S(\rho_{\theta A}\rho_{\theta S} - \rho_{AS}) + \sigma_S^2(1 - \rho_{\theta S}^2)}{(1 - \rho_{AS}^2)\sigma_A^2\sigma_S^2}}\Bigg] \tag{47}$$

To ensure the solutions exist and $\overline{v}^{I_0}, \overline{v}^{I_1} \geq 0$, the following conditions have to be satisfied

$$b_1^2\sigma_A^2 - 2b_1\sigma_A\sigma_S\rho_{AS} + \sigma_S^2 - (b_1\sigma_A\rho_{\theta A} - \sigma_S\rho_{AS})^2 > 0 \tag{48}$$

$$1 - \rho_{AS}^2 - \rho_{\theta S}^2 - \rho_{\theta A}^2 + 2\rho_{AS}\rho_{\theta A}\rho_{\theta S} \leq 0 \tag{49}$$

From Equation (47), one could see that with $b_1 \neq 0$, the signal not only helps to understand the predictability of past realized return (through ρ_{AS}), but also provides an extra source of information to learn about θ (through $\rho_{\theta S}$). Therefore, $\overline{v}^{\mathcal{I}_1}|_{b_1 \neq 0} \leq \overline{v}^{\mathcal{I}_1}|_{b_1 = 0}$. To prove the proposition, we only need to show $\overline{v}^{\mathcal{I}_0} \geq \overline{v}^{\mathcal{I}_1}|_{b_1 = 0}$. With $b_1 = 0, dS_t = \sigma_S dZ_{St}$. The variance of the estimation for the 1-signal case becomes

$$\overline{v}^{\mathcal{I}_1}|_{b_1=0} = \sigma_A\sigma_\theta\Big(\rho_{AS}\rho_{\theta S} - \rho_{\theta A} - \sqrt{(1 - \rho_{\theta S}^2)(1 - \rho_{AS}^2)}\Big) \tag{50}$$

As $\rho_{AS}\rho_{\theta S} < 1$, comparing with (47), one could then show

$$\overline{v}^{\mathcal{I}_1}|_{b_1 = 0} = \overline{v}^{\mathcal{I}_0} - \sigma_A\sigma_\theta[1 - \rho_{AS}\rho_{\theta S} + \sqrt{(1 - \rho_{\theta S}^2)(1 - \rho_{AS}^2)}] \leq \overline{v}^{\mathcal{I}_0}$$

$$\overline{v}^{\mathcal{I}_1}|_{b_1 \neq 0} \leq \overline{v}^{\mathcal{I}_1}|_{b_1 = 0} \leq \overline{v}^{\mathcal{I}_0} \tag{51}$$

■

Proof of Propositions 3 – 5

The optimization for the firm with information set $\mathcal{I}_i$ for $i \in \{0, 1, 2\}$ constitutes an objective function (16) and the intertemporal budget constrain (15) ((17) for the perfect information case). By the dynamic programming presented in Dixit and Pindyck (1994), one can write down the corresponding Hamilton–Jacobi–Bellman (HJB) equation as

$$\underbrace{\rho J_i(W,m)}_{\text{required rate of return}} = \max_{\{c_t,\omega_t\}} \underbrace{u(c)}_{\text{instantaneous utility flow}} + \underbrace{[(r + (m - \alpha - r)\omega)W - c]J_{iW} + \frac{1}{2}\sigma_A^2\omega^2 W^2 J_{iWW}}_{\text{evolution of wealth}}$$
$$+ \underbrace{(a_0 + a_1 m)J_{im} + \frac{1}{2}\Psi_i J_{imm}}_{\text{evoluation of estimation}} + \underbrace{(\sigma_{\theta A} + \overline{v}^{\mathcal{I}_i})\omega W J_{iWm}}_{\text{compound effect of wealth and estimation}} \tag{52}$$

where Ψ_i represents the information gain (reduction in the estimation error)

$$\Psi_0 \equiv \left(\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_1}}{\sigma_A}\right)^2, \;\; \Psi_1 \equiv \xi_A^2 + \xi_S^2 + 2\rho_{AS}\xi_A\xi_S \text{ and } \Psi_2 \equiv \sigma_\theta^2 \tag{53}$$

with ξ_A and ξ_S defined in (8). The underscripts represent the partial derivative of J_i with respect to the state variables. The Ψ_2 reflects the fact that with perfect learning, the agent is able to identify the true value θ_t at each point of time. The information gain at each point of time is $\Psi_2 = \sigma_\theta^2$, which exactly offsets the total noise and makes $v^{\mathcal{I}_2} = 0$. Therefore, for the perfect information case, $m_t^{\mathcal{I}_2} = \theta_t$ for all t.

The first order conditions are given by

$$c^\star = (J_{iW})^{-\frac{1}{\gamma}} \tag{54}$$
$$\omega^\star = -\frac{m - \alpha - r}{\sigma_A^2}\frac{J_{iW}}{WJ_{iWW}} - \frac{\sigma_{\theta A} + \overline{v}_i^{\mathcal{I}}}{\sigma_A^2}\frac{J_{imW}}{WJ_{iWW}} \tag{55}$$

Substituting the conjecture (18) into the first-order conditions and the HJB equation (52) shows the main arguments in the propositions.

The optimality also requires the indirect utility function $J_i(W,m)$ to be concave in both W and m, that is, $J_{iWW} < 0$ and $J_{imm} < 0$, which is verified numerically. The concave and continuous utility function (power utility function) also suggests a diminishing marginal utility with respect to both W and m, which means that $\lim_{m\to\pm\infty} J_{im} = J_{iW} = 0$. Plugging the conjecture shows the boundary conditions in the propostions. Note that although the main discussion of the paper is based on θ_t that follows an arithmetic Brownian motion, i.e. $a_1 = 0$, the proof also holds for the case where θ_t follows a mean-reverting process, i.e. $a_1 < 0$.

As another useful benchmark, I also present the main results for the Merton (1969) case where the θ is constant. With constant drift, the optimal consumption and investment policy are: $c_t^M = W_t(1-\gamma)^{-\frac{1}{\gamma}} g_M^{-\frac{1}{\gamma}}$ and $\omega^M = \frac{\theta - r}{\gamma \sigma_A^2}$. g_M is a constant that satisfies the following

$$\Big(r(1-\gamma) - \rho + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{(\theta - r)^2}{\sigma_A^2}\Big) g_M(\theta) + \frac{\gamma}{1-\gamma}(1-\gamma)^{1-\frac{1}{\gamma}} g_M(\theta)^{1-\frac{1}{\gamma}} = 0 \tag{56}$$

■

Derivation of the critical α

Take α_{01} as an example. α_{01} is defined to be the cutoff cost that makes the firm indifferent between keeping the endowed information set $\mathcal{I}_0$ and having the expanded information set $\mathcal{I}_1$. In other words, with α_{01}, $J_0(m, W_t) = J_1(m, W_t; \alpha_{01})$. Therefore, given W_t and m, $g_0(m) = g_(m, \alpha_{01})$. Substituting $g_0(m)$ into (24) (the ODE for $g_1(m, \alpha_{01})$), together with (21) (the ODE for $g_0(m)$), shows

$$\begin{aligned}
&\Big(r(1-\gamma) - \rho + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{(m - \alpha_{01} - r)^2}{\sigma_A^2}\Big) g_0(m) + \frac{\gamma}{1-\gamma}(1-\gamma)^{1-\frac{1}{\gamma}} g_0(m)^{1-\frac{1}{\gamma}} \\
&+ \Big(a_0 + a_1 m + \frac{1-\gamma}{\gamma}\frac{(m - \alpha_{01} - r)}{\sigma_A}\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_1}}{\sigma_A}\Big) g_0'(m) + \frac{1}{2}\Big(\xi_A^2 + \xi_S^2 + 2\rho_{AS}\xi_A\xi_S\Big) g_0''(m) \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\Big(\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_1}}{\sigma_A}\Big)^2 \frac{g_0'(m)^2}{g_0(m)} \\
&= \Big(r(1-\gamma) - \rho + \frac{1}{2}\frac{1-\gamma}{\gamma}\frac{(m - r)^2}{\sigma_A^2}\Big) g_0(m) + \frac{\gamma}{1-\gamma}(1-\gamma)^{1-\frac{1}{\gamma}} g_0(m)^{1-\frac{1}{\gamma}} \\
&+ \Big(a_0 + a_1 m + \frac{1-\gamma}{\gamma}\frac{m - r}{\sigma_A}\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_0}}{\sigma_A}\Big) g_0'(m) + \frac{1}{2}\Big(\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_0}}{\sigma_A}\Big)^2 g_0''(m) \\
&+ \frac{1}{2}\frac{1-\gamma}{\gamma}\Big(\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_0}}{\sigma_A}\Big)^2 \frac{g_0'(m)^2}{g_0(m)}
\end{aligned}$$

Simplifing the above expression shows Equationn (36). α_{01} is therefore the solution to a quadratic function and can be written as

$$\alpha_{01} = \frac{(\sigma_{\theta A} + \overline{v}^{\mathcal{I}_1})g_0'(m) - (r - m_t)g_0(m) \pm \sqrt{\Delta}}{g_0(m)} \text{ with} \tag{57}$$

$$\Delta = [(\sigma_{\theta A} + \overline{v}^{\mathcal{I}_1})g_0'(m) - (r - m_t)g_0(m)]^2 - 2g_0(m)\times$$
$$\Bigg[(m - r)(\overline{v}^{\mathcal{I}_0} - \overline{v}^{\mathcal{I}_1})g_0'(m) + \frac{1}{2}\sigma_A^2\frac{\gamma}{1-\gamma}\Big[\Big(\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_0}}{\sigma_A}\Big)^2 - (\xi_A + \xi_S + 2\rho_{AS}\xi_A\xi_S)\Big]g_0''(m)$$
$$+ \frac{1}{2}\Big[2\sigma_{\theta A}(\overline{v}^{\mathcal{I}_0} - \overline{v}^{\mathcal{I}_1}) + (\overline{v}^{\mathcal{I}_0})^2 - (\overline{v}^{\mathcal{I}_1})^2\Big]\frac{g_0'(m)^2}{g_0(m)}\Bigg] > 0$$

For all m, the quadratic function has two roots, one positive and one negative. In the numerical analysis, I choose $\alpha_{01}(m)$ such that it is positive for a long position and negative for a short position to ensure that it is an expense, not an income. This means that $\alpha_{01}(m)$ satisfies the following

$$sign(\omega_1^*(m, \alpha = 0)) = sign(\omega_1^*(m, \alpha)) \tag{58}$$

With the same method, one can derive the following quadratic equation for α_{02}

$$\frac{1}{2}g_0(m)\alpha_{02}^2 + [(r - m)g_0(m) - \sigma_{\theta A}g_0'(m)]\alpha_{02} \tag{59}$$
$$= (m - r)\overline{v}^{\mathcal{I}_0}g_0'(m) - \frac{1}{2}\sigma_A^2\gamma(\sigma_\theta^2 - (\frac{\sigma_{\theta A} + \overline{v}^{\mathcal{I}_0}}{\sigma_A})^2)g_0''(m) + \frac{1}{2}(2\sigma_{\theta A}\overline{v}^{\mathcal{I}_0} + \overline{v}^{\mathcal{I}_0^2})\frac{g_0'(m)^2}{g_0(m)}$$

and similarly for α_{12}.

Proof of Corollary 2 and 3

To study the total effect of changing estimation error on the payout, $\frac{\partial c}{\partial \overline{v}}$, we can first look at the total derivative of the indirect utility function J with respect to $\overline{v}$ for a given level of $\overline{J}$

$$\frac{dJ}{d\overline{v}}|_{J=\overline{J}} = \frac{\partial g}{\partial \overline{v}}W^{1-\gamma} + g(1-\gamma)W^{-\gamma}\frac{\partial W}{\partial \overline{v}} = 0 \Rightarrow \frac{\partial W}{\partial \overline{v}}|_{\overline{J}} = -\frac{1}{(1-\gamma)g}\frac{\partial g}{\partial \overline{v}}W < 0 \tag{60}$$

Therefore, the payout substitution effect from change in the estimation error is given by

$$\frac{\partial c}{\partial \overline{v}}|_{\overline{J}} = \frac{\partial}{\partial \overline{v}}(((1-\gamma)g(m))^{-\frac{1}{\gamma}})\,W + ((1-\gamma)g(m))^{-\frac{1}{\gamma}}\frac{\partial W}{\partial \overline{v}}|_{\overline{J}}$$

$$= -((1-\gamma)g(m))^{-\frac{1}{\gamma}}\frac{\partial g}{\partial \overline{v}}\frac{W}{g}\frac{1}{\gamma(1-\gamma)} > 0 \tag{61}$$

The wealth effect is negative:

$$\frac{dc}{d\overline{v}} - \frac{\partial c}{\partial \overline{v}}|_{\overline{J}} = \frac{\partial}{\partial}((1-\gamma)g(m))^{-\frac{1}{\gamma}}W + ((1-\gamma)g(m))^{-\frac{1}{\gamma}}\frac{\partial g}{\partial \overline{v}}\frac{W}{g}\frac{1}{\gamma(1-\gamma)} \tag{62}$$

$$= ((1-\gamma)g(m))^{-\frac{1}{\gamma}}\frac{W}{\gamma g}\frac{\partial g}{\partial \overline{v}}\frac{\gamma}{1-\gamma} < 0 \tag{63}$$

All else equal, the total effect on payout and aggregate welfare

$$\frac{dc}{d\overline{v}} = \frac{\partial}{\partial \overline{v}}((1-\gamma)g(m))^{-\frac{1}{\gamma}}W < 0 \quad \text{and} \quad \frac{dJ}{d\overline{v}} = \frac{\partial g}{\partial \overline{v}}W^{1-\gamma} < 0 \tag{64}$$

is negative, i.e increases in estimation error reduce the payout yield and investors' welfare. As $m \to \pm\infty$, the behavior of the investment, payout and investors' welfare follows directly from the boundary conditions.

■

References

Acharya, V. V., and B. M. Lambrecht (2015): "A theory of income smoothing when insiders know more than outsiders," *Review of Financial Studies*, 28(9), 2534–2574.

Antweiler, W., and M. Z. Frank (2004): "Is all that talk just noise? The information content of internet stock message boards," *Journal of Finance*, 59(3), 1259–1294.

Barberis, N. (2000): "Investing for the long run when returns are predictable," *Journal of Finance*, 55(1), 225–264.

Barlevy, G., and P. Veronesi (2000): "Information acquisition in financial markets," *Review of Economic Studies*, 67(1), 79–90.

Bhattacharya, U., N. Galpin, R. Ray, and X. Yu (2009): "The role of the media in the internet IPO bubble," *Journal of Financial and Quantitative Analysis*, 44(3), 657–682.

Brandt, M. W., A. Goyal, P. Santa-Clara, and J. R. Stroud (2005): "A simulation approach to dynamic portfolio choice with an application to learning about return predictability," *Review of Financial Studies*, 18(3), 831–873.

Brennan, M. J. (1998): "The role of learning in dynamic portfolio decisions," *Review of Finance*, 1(3), 295–306.

Campo, S., E. Guerre, I. Perrigne, and Q. Vuong (2011): "Semiparametric estimation of first-price auctions with risk-averse bidders," *Review of Economic Studies*, 78(1), 112–147.

Chen, D., Y. Ma, X. Martin, and R. Michaely (2020): "On the fast track: Information acquisition costs and information production," Working paper, Swiss Finance Institute.

Chen, H., P. De, Y. J. Hu, and B.-H. Hwang (2014): "Wisdom of crowds: The value of stock opinions transmitted through social media," *Review of Financial Studies*, 27(5), 1367–1403.

Chen, H., P. Gompers, A. Kovner, and J. Lerner (2010): "Buy local? The geography of venture capital," *Journal of Urban Economics*, 67(1), 90–102.

Chetty, R. (2006): "A new method of estimating risk aversion," *American Economic Review*, 96(5), 1821–1834.

Coval, J. D., and T. J. Moskowitz (1999): "Home bias at home: Local equity preference in domestic portfolios," *Journal of Finance*, 54(6), 2045–2073.

——— (2001): "The geography of investment: Informed trading and asset prices," *Journal of Political Economy*, 109(4), 811–841.

Curran, J. (2019): "Financial data service providers," Industry (specialized) report, IBIS World.

Daley, B., T. Geelen, and B. Green (2020): "Due diligence," Working paper, SSRN.

Dangl, T., Y. Wu, and J. Zechner (2008): "Market discipline and internal governance in the mutual fund industry," *Review of Financial Studies*, 21(5), 2307–2343.

Décamps, J.-P., T. Mariotti, and S. Villeneuve (2005): "Investment timing under incomplete information," *Mathematics of Operations Research*, 30(2), 472–500.

DeMarzo, P. M., and Y. Sannikov (2017): "Learning, termination, and payout policy in dynamic incentive contracts," *The Review of Economic Studies*, 84(1), 182–236.

Detemple, J. B. (1986): "Asset pricing in a production economy with incomplete information," *Journal of Finance*, 41(2), 383–391.

Dixit, A., and R. Pindyck (1994): *Investment under uncertainty*. Princeton University Press, Princeton, New Jersey.

Fama, E. F., and K. R. French (2000): "Trading is hazardous to your wealth: The common stock investment performance of individual investors," *Journal of Finance*, 65(5), 773–806.

Feldman, D. (1992): "Logarithmic preferences, myopic decisions, and incomplete information," *Journal of Financial and Quantitative Analysis*, 27(4), 619–629.

Gao, M., and J. Huang (2020): "Informing the market: The effect of modern information technologies on information production," *Review of Financial Studies*, 33(4), 1367–1411.

Garlappi, L., R. Uppal, and T. Wang (2007): "Portfolio selection with parameter and model uncertainty: A multi-prior approach," *Review of Financial Studies*, 20(1), 41–81.

Gennotte, G. (1986): "Optimal portfolio choice under incomplete information," *Journal of Finance*, 41(3), 733–746.

Giroud, X. (2013): "Proximity and investment: Evidence from plant-level data," *Quarterly Journal of Economics*, 128(2), 861–915.

Glode, V. (2011): "Why mutual funds "underperform"," *Journal of Financial Economics*, 99(3), 546–559.

Goldstein, I., and L. Yang (2017): "Information disclosure in financial markets," *Annual Review of Financial Economics*, 9, 101–125.

Goldstein, I., S. Yang, and L. Zuo (2020): "The real effects of modern information technologies," Working paper, National Bureau of Economic Research.

Grenadier, S. R. (1999): "Information revelation through option exercise," *Review of Financial Studies*, 12(1), 95–129.

Grenadier, S. R., and A. Malenko (2010): “A Bayesian approach to real options: The case of distinguishing between temporary and permanent shocks,” *Journal of Finance*, 65(5), 1949–1986.

——— (2011): “Real options signaling games with applications to corporate finance,” *Review of Financial Studies*, 24(12), 3993–4036.

He, Z., B. Wei, J. Yu, and F. Gao (2017): “Optimal long-term contracting with learning,” *Review of Financial Studies*, 30(6), 2006–2065.

Hsu, Y.-W., and B. M. Lambrecht (2007): “Preemptive patenting under uncertainty and asymmetric information,” *Annals of Operations Research*, 151(1), 5–28.

Huang, A. H., A. Y. Zang, and R. Zheng (2014): “Evidence on the information content of text in analyst reports,” *The Accounting Review*, 89(6), 2151–2180.

Iliev, P., and M. Lowry (2015): “Are mutual funds active voters?,” *Review of Financial Studies*, 28(2), 446–485.

Jaffe, J. F., and J. M. Mahoney (1999): “The performance of investment newsletters,” *Journal of Financial Economics*, 53(2), 289–307.

Jagannathan, R., and B. Liu (2019): “Dividend dynamics, learning, and expected stock index returns,” *Journal of Finance*, 74(1), 401–448.

Jagannathan, R., B. Liu, and J. Zhang (2019): “Corrigendum for dividend dynamics, learning, and expected stock index returns,” *Journal of Finance*, 74(4), 2107–2116.

Jegadeesh, N., J. Kim, S. D. Krische, and C. M. Lee (2004): “Analyzing the analysts: When do recommendations add value?,” *Journal of Finance*, 59(3), 1083–1124.

Jensen, M. C. (1968): “The performance of mutual funds in the period 1945-1964,” *Journal of Finance*, 23(2), 389–416.

Johannes, M., A. Korteweg, and N. Polson (2014): “Sequential learning, predictability, and optimal portfolio returns,” *Journal of Finance*, 69(2), 611–644.

Kacperczyk, M., S. Van Nieuwerburgh, and L. Veldkamp (2014): “Time-varying fund manager skill,” *Journal of Finance*, 69(4), 1455–1484.

——— (2016): "A rational theory of mutual funds' attention allocation," *Econometrica*, 84(2), 571–626.

Kan, R., and G. Zhou (2007): "Optimal portfolio choice with parameter uncertainty," *Journal of Financial and Quantitative Analysis*, 42(3), 621–656.

Klein, M. (2009): "Comment on "Investment timing under incomplete information"," *Mathematics of Operations Research*, 34(1), 249–254.

Lambrecht, B., and W. Perraudin (2003): "Real options and preemption under incomplete information," *Journal of Economic Dynamics and Control*, 27(4), 619–643.

Larcker, D. F., A. L. McCall, and G. Ormazabal (2015): "Outsourcing shareholder voting to proxy advisory firms," *The Journal of Law and Economics*, 58(1), 173–204.

Li, G. (2005): "Information quality, learning, and stock market returns," *Journal of Financial and Quantitative Analysis*, 40(3), 595–620.

Liptser, R. S., and A. N. Shiryaev (2001a): *Statistics of random processes: I. General theory.* Springer, 2nd, rev. and expanded ed. edn.

——— (2001b): *Statistics of random processes: II. Applications.* Springer, 2nd, rev. and expanded ed. edn.

Malenko, N., and Y. Shen (2016): "The role of proxy advisory firms: Evidence from a regression-discontinuity design," *Review of Financial Studies*, 29(12), 3394–3427.

Merton, R. C. (1969): "Lifetime portfolio selection under uncertainty: The continuous-time case," *The Review of Economics and Statistics*, 51(3), 247–257.

——— (1971): "Optimum consumption and portfolio rules in a continuous-time model," *Journal of Economic Theory*, 3(4), 373–413.

——— (1973): "An intertemporal capital asset pricing model," *Econometrica*, 41(5), 867–887.

Metrick, A. (1999): "Performance evaluation with transactions data: The stock selection of investment newsletters," *Journal of Finance*, 54(5), 1743–1775.

Morgan, J., and P. C. Stocken (2003): "An analysis of stock recommendations," *Review of Economic Studies*, 34(1), 183–203.

Moskowitz, T. J. (2000): "Mutual fund performance: An empirical decomposition into stock-picking talent, style, transactions costs, and expenses: Discussion," *Journal of Finance*, 55(4), 1695–1703.

Pastor, L., and P. Veronesi (2009): "Learning in financial markets," *Annual Review of Financial Economics*, 1(1), 361–381.

Peng, L. (2005): "Learning with information capacity constraints," *Journal of Financial and Quantitative Analysis*, 40(2), 307–329.

Veronesi, P. (2000): "How does information quality affect stock returns?," *Journal of Finance*, 55(2), 807–837.

Wang, J. (1993): "A model of intertemporal asset prices under asymmetric information," *Review of Economic Studies*, 60(2), 249–282.

Wermers, R. (2000): "Mutual fund performance: An empirical decomposition into stock-picking talent, style, transactions costs, and expenses," *Journal of Finance*, 55(4), 1655–1695.

Williams, J. T. (1977): "Capital asset prices with heterogeneous beliefs," *Journal of Financial Economics*, 5(2), 219–239.

Xia, Y. (2001): "Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation," *Journal of Finance*, 56(1), 205–246.

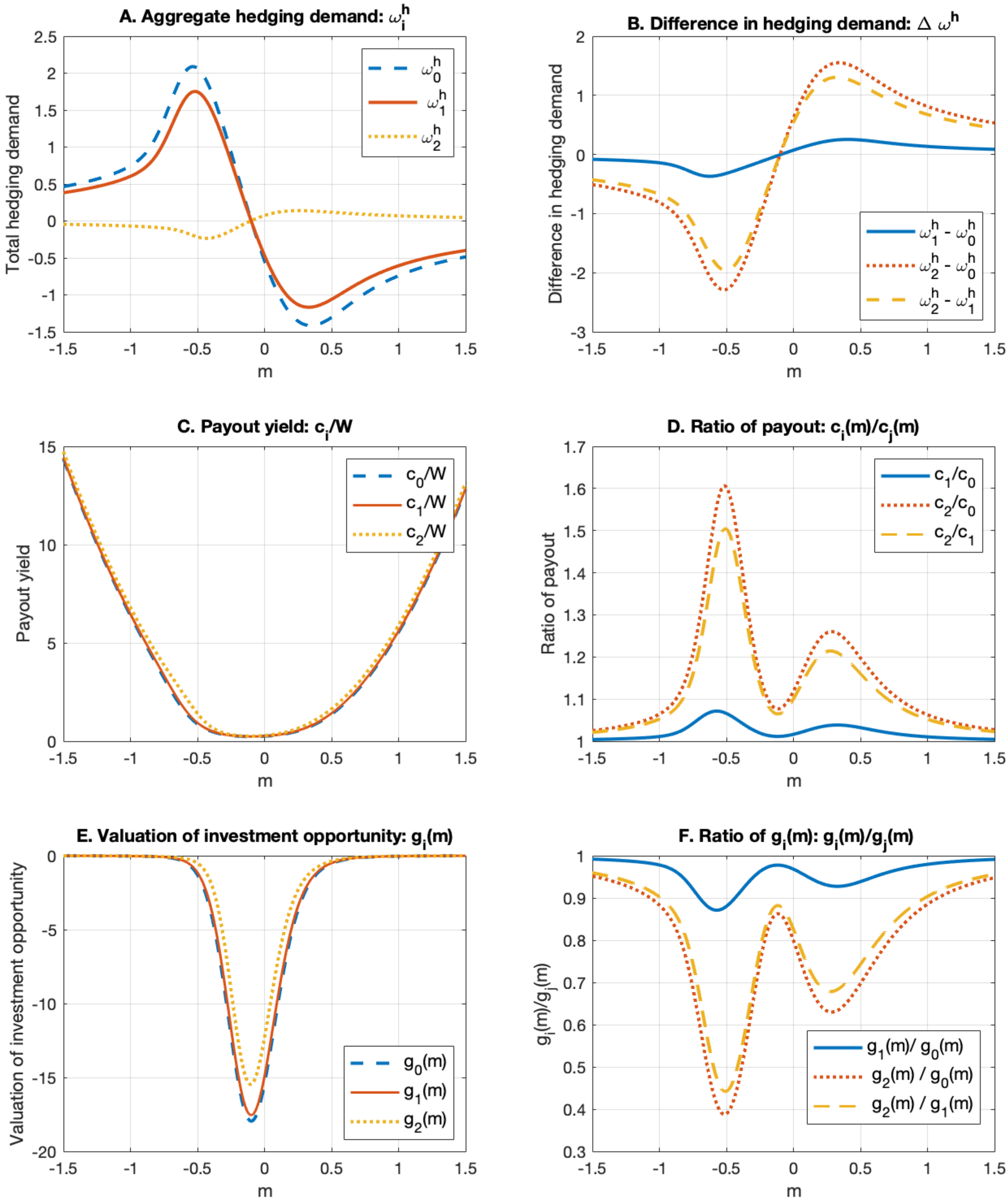

Figure 1: Optimal policies and welfare

Panels A, C, E, plot respectively the aggregate hedging demand(ω_i^h), payout yield (c_i/W), and valuation of the investment opportunity $g_i(m)$ as a function of the estimated rate of return m. The dashed lines represent the 0-signal case, whereas the solid lines represent the 1-signal case and the dotted lines refer to the perfect information case. The remaining panels present the comparisons between different information sets. Panels B, D, F plot respectively the difference in the aggregate hedging demand ($\Delta\omega^h$), the ratio of payout yield ($c_i(m)/c_j(m)$), and the ratio of the indirect utility function given total net worth ($g_i(m)/g_j(m)$) for different levels of estimated rate of return (m). The solid line represents the comparison between the 1-signal and 0-signal case, while the dotted lines represent the comparison between the perfect information and the 0-signal case and the dashed lines demonstrate the comparison between the perfect information and the 1-signal case. Note that the comparison of aggregate investment is expressed in an absolute term as the aggregate hedging demand can be zero, which makes the ratios spike up and the graph uninformative. The figure is computed for the following parameter values (as in Table 1): $\sigma_A = 0.14, a_0 = 0.09, \sigma_\theta = 0.05, b_1 = 0.084, \sigma_S = 0.15, \rho_{\theta A} = -0.1, \rho_{\theta S} = 0.5, \rho_{AS} = 0.01, r = 0.05, \rho = 0.1$ and $\gamma = 2$.

Table 2: Decomposition of the optimal investment policy

	Magnitude						Percentage ω_i^k/ω_i (in %)					
m	-1.50	-0.50	-0.05	0.00	0.05	1.50	-1.50	-0.50	-0.05	0.00	0.05	1.50
Panel A: Myopic portfolio allocation ω_i^m												
0-Signal	-39.541	-14.031	-2.554	-1.276	0.000	36.990	101.190	117.260	90.259	70.297	0.000	101.330
1-Signal	-39.541	-14.031	-2.554	-1.276	0.000	36.990	100.976	114.206	91.341	73.197	0.000	101.091
PI	-39.541	-14.031	-2.554	-1.276	0.000	36.990	99.886	98.483	101.649	106.155	0.000	99.872
Panel B: Opportunity-uncertainty hedging demand ω_i^u												
0-Signal	-0.047	-0.207	0.028	0.054	0.077	0.049	0.119	1.726	-0.974	-2.970	-10.000	0.133
1-Signal	-0.046	-0.211	0.029	0.057	0.081	0.048	0.118	1.721	-1.049	-3.247	-12.113	0.132
PI	-0.045	-0.216	0.041	0.074	0.100	0.047	0.114	1.517	-1.649	-6.155	100.000	0.128
Panel C: Estimation-error hedging demand ω_i^v												
0-Signal	0.512	2.272	-0.303	-0.593	-0.852	-0.534	-1.309	-18.986	10.715	32.674	110.000	-1.464
1-Signal	0.428	1.957	-0.271	-0.524	-0.747	-0.447	-1.094	-15.927	9.708	30.050	112.113	-1.223
PI	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Panel D: Total hedging demand ω_i^h												
0-Signal	0.465	2.065	-0.276	-0.539	-0.775	-0.486	-1.190	-17.260	9.741	29.703	100.000	-1.330
1-Signal	0.382	1.745	-0.242	-0.467	-0.666	-0.399	-0.976	-14.206	8.659	26.803	100.000	-1.091
PI	-0.045	-0.216	0.041	0.074	0.100	0.047	0.114	1.517	-1.649	-6.155	100.000	0.128
Panel E: Optimal investment policy ω_i												
0-Signal	-39.076	-11.966	-2.829	-1.815	-0.775	36.504	100.000	100.000	100.000	100.000	100.000	100.000
1-Signal	-39.159	-12.286	-2.796	-1.743	-0.666	36.591	100.000	100.000	100.000	100.000	100.000	100.000
PI	-39.587	-14.247	-2.512	-1.202	0.100	37.037	100.000	100.000	100.000	100.000	100.000	100.000

Panels A, B, C, D and E show the numerical result for the myopic portfolio allocation (ω_i^m), the opportunity-uncertainty hedging demand (ω_i^u), the estimation-error hedging demand (ω_i^v), the total hedging demand (ω_i^h) and the aggregate risky investment (ω), respectively, in different values of estimation m. The left hand-side shows the magnitude of each component, while the right hand-side shows the relative importance of the component (i.e. the corresponding percentage in the aggregate investment (ω_i^k/ω_i). The table is computed for the following parameter values (as in Table 1): $\sigma_A = 0.14, a_0 = 0.09, \sigma_\theta = 0.05, b_1 = 0.084, \sigma_S = 0.15, \rho_{\theta A} = -0.1, \rho_{\theta S} = 0.5, \rho_{AS} = 0.01, r = 0.05, \rho = 0.1$ and $\gamma = 2$

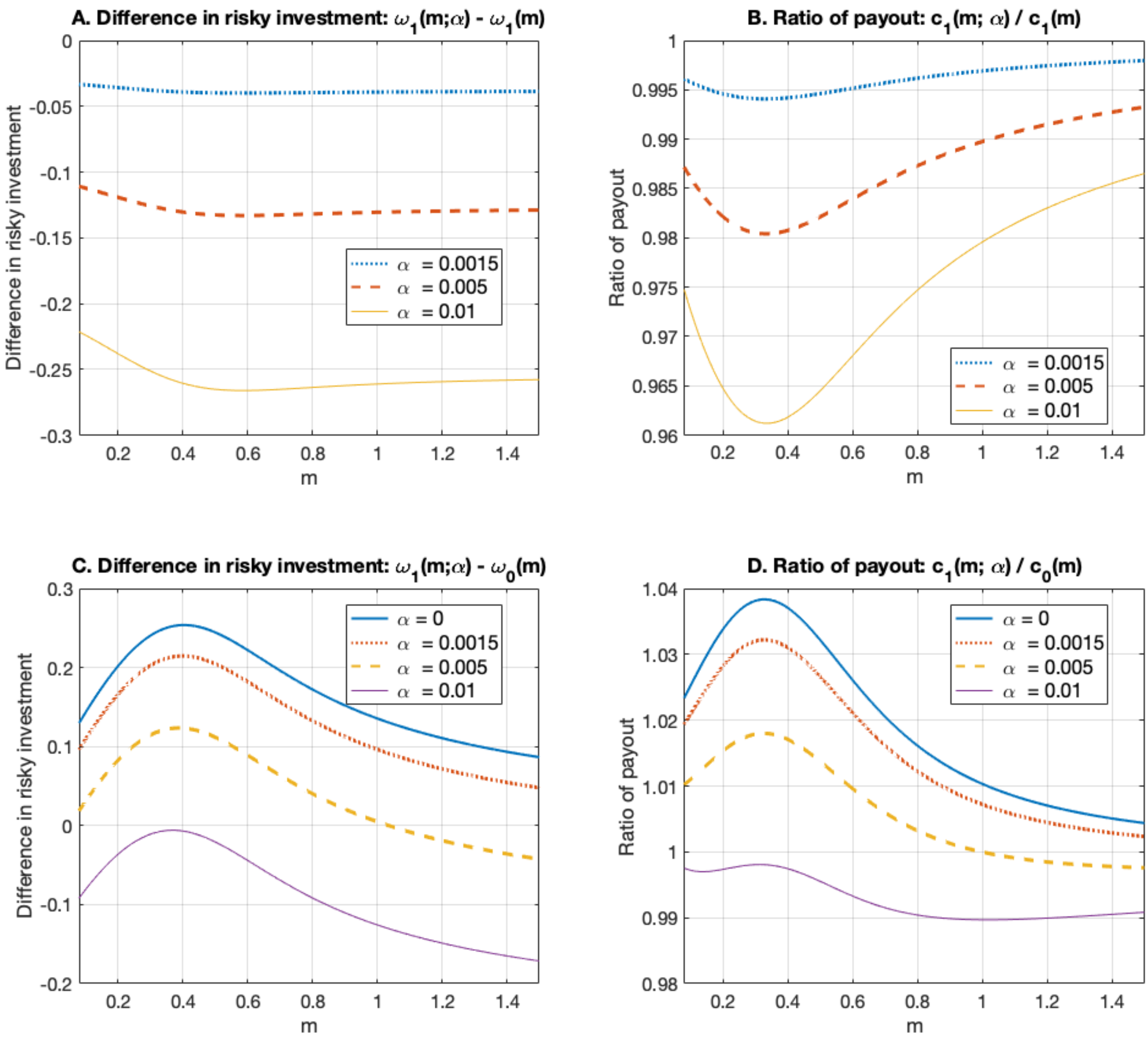

Figure 2: Distortion effect of information cost on the investment and payout policies

The graphs plot the distortion effect of information cost on the investment and payout policy for different estimated rates of return m and different levels of cost α. The top panels compare the optimal policies between the costly ($\alpha \neq 0$) case and the costless ($\alpha = 0$) case: Panel A plots the difference in the risky investment $\omega_1(m;\alpha) - \omega_1(m)$, while Panel B plots the ratio of the payout yield $c_1(m;\alpha)/c_1(m;\alpha)$. The bottom panels compare the optimal policies between the costly 1-signal case with the 0-signal case: Panel C plots the comparison of the risky investment $\omega_1(m;\alpha) - \omega_0(m)$ and Panel D shows the comparison of the payout $c_1(m;\alpha)/c_0(m)$. The benchmark case is $\alpha = 0$ (thick solid line). The information cost (α) takes the value of: 0.0015 (dotted line), 0.005 (dashed line), 0.01 (solid line).

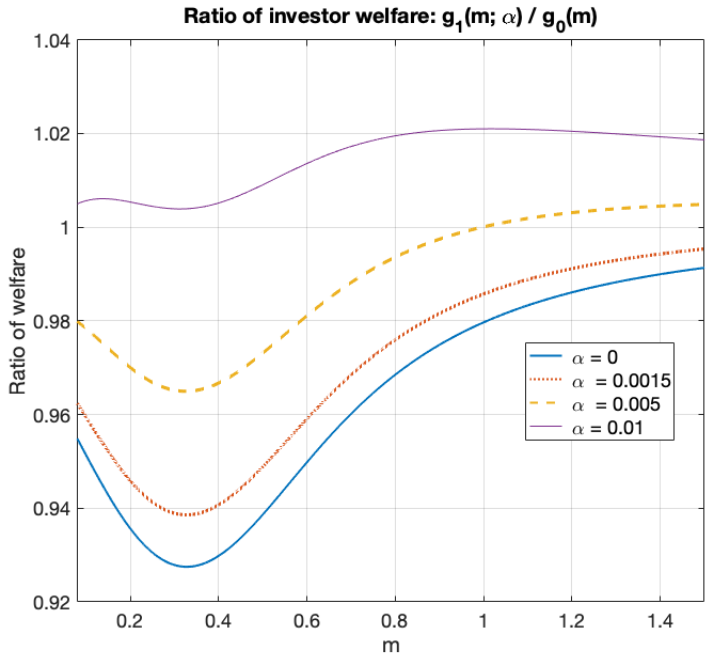

Figure 3: Distortion effect of information cost on the welfare

The graph plots the distortion of the information cost on the firm's welfare by comparing the investors' indirect utility function under the costly 1-signal case and the 0-signal case. It plots $g_1(m;\alpha)/g_0(m)$ as a function of the estimated rate of return m for different levels of cost α. The benchmark case is $\alpha = 0$ (thick solid line). The information cost (α) takes the value of: 0.0015 (dotted line), 0.005 (dashed line), 0.01 (solid line).

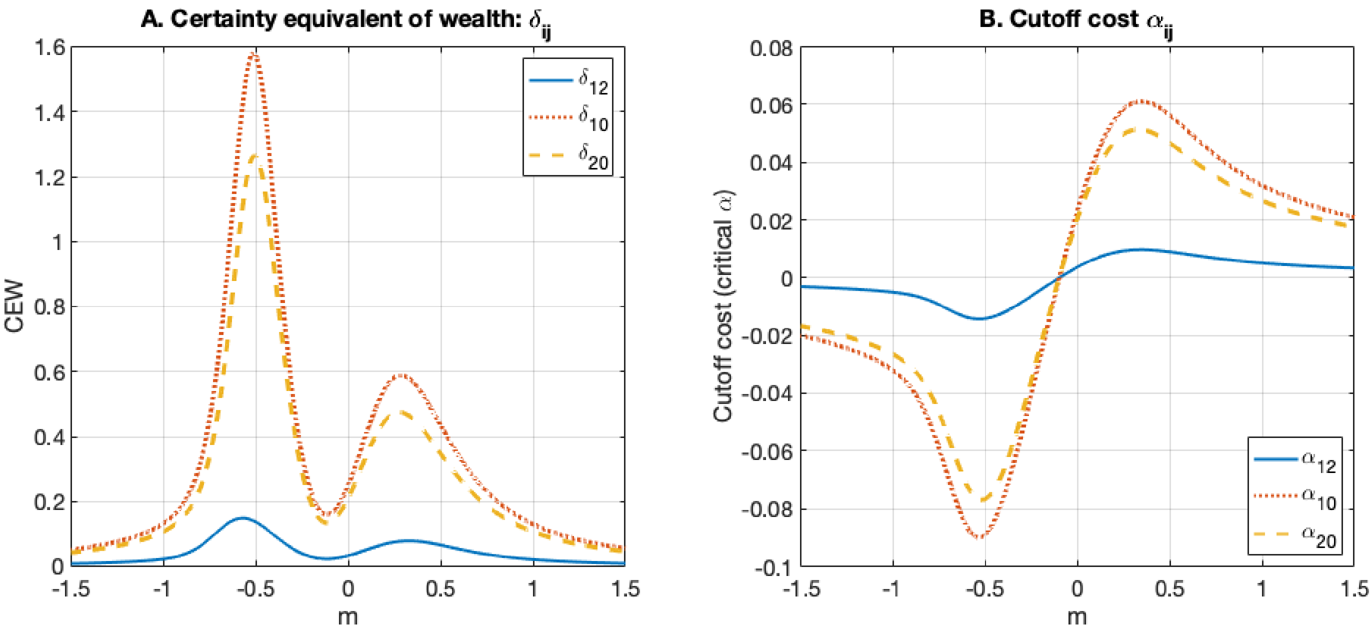

Figure 4: Value of information

The graphs show two measures of the information value for different estimated rates of return m. Panel A plots the certainty equivalent of wealth $\delta_{ij}(m)$, while Panel B plots the cutoff cost boundary $\alpha_{ij}(m)$. The graphs are computed for the following parameter values (as in Table 1): $\sigma_A = 0.14, a_0 = 0.09, \sigma_\theta = 0.05, b_1 = 0.084, \sigma_S = 0.15, \rho_{\theta A} = -0.1, \rho_{\theta S} = 0.5, \rho_{AS} = 0.01, r = 0.05, \rho = 0.1$ and $\gamma = 2$.

Table 3: The cutoff cost boundary and changes in investors' welfare

	m							
	0.07	0.10	0.12	0.15	0.20	0.40	1.00	1.50
m-r	0.02	0.05	0.07	0.10	0.15	0.35	0.95	1.45
	Panel A: Cutoff cost boundary $\alpha_{ij}(m)$							
$\alpha_{01}(m)$	0.006	0.007	0.007	0.008	0.009	0.010	0.005	0.003
$\alpha_{02}(m)$	0.037	0.042	0.045	0.049	0.055	0.060	0.032	0.021
$\alpha_{12}(m)$	0.032	0.037	0.039	0.043	0.047	0.050	0.027	0.018
	Panel B: Improvement in welfare $\frac{g_j(m)}{g_i(m)} - 1$							
$g_1(m)/g_0(m) - 1$	-0.043	-0.049	-0.052	-0.057	-0.064	-0.070	-0.020	-0.009
$g_2(m)/g_0(m) - 1$	-0.266	-0.292	-0.308	-0.329	-0.355	-0.347	-0.115	-0.052
$g_2(m)/g_1(m) - 1$	-0.233	-0.256	-0.270	-0.288	-0.310	-0.298	-0.096	-0.044

The table shows the estimation of the cutoff cost boundary from the model for different levels of expected rate of return m. $m-r$ is the expected rate of return net of the risk-free rate. Panel A shows the cutoff cost $\alpha_{ij}(m)$, which represents the maximum cost (expressed as a proportion of total investment) the firm is willing to bear in order to replace the information set $\mathcal{I}_i$ with $\mathcal{I}_j$. Panel B shows the corresponding changes in the investors' welfare $g_j(m)/g_i(m)-1$. The table is computed for the following parameter values (as in Table 1): $\sigma_A = 0.14, a_0 = 0.09, \sigma_\theta = 0.05, b_1 = 0.084, \sigma_S = 0.15, \rho_{\theta A} = -0.1, \rho_{\theta S} = 0.5, \rho_{AS} = 0.01, r = 0.05, \rho = 0.1$ and $\gamma = 2$.